Mass Estimation Using the Virial Theorem

The big use of the virial theorem is to determine the mass of a *static* assembly of point masses – for example, the total mass of a globular cluster of (perhaps) 10^5 stars, or a cluster of 10^3 galaxies. In the simplest application of the virial theorem, we assume spherical symmetry and that all the objects are of the same mass, m. There are refinements of the theory where these assumptions are dropped, which makes the manipulations more difficult. We use the virial theorem in the form

$$GM = \overline{v^2}\, R_e \quad ,$$

where for N equal-mass objects

$$\begin{aligned} \overline{v^2} &= \frac{\sum m_i v_i^2}{\sum m_i} \\ &= \frac{1}{N} \sum v_i^2 \\ \frac{1}{R_e} &= \frac{\sum_{i,j} m_i m_j / r_{ij}}{\left(\sum m_i\right)^2} \\ &= \frac{1}{N^2} \sum_{i,j \text{ pairs}} \frac{1}{r_{ij}} \quad . \end{aligned}$$

Then to get the total mass, M, we must measure the mean square velocity of galaxies in the cluster, $\overline{v^2}$, the velocity dispersion; and the harmonic mean radius of galaxies in the cluster, R_e, the "effective radius."

How do we get these from what we see?

a) *Measurements of $\sqrt{\overline{v^2}}$, the velocity dispersion*

We measure velocities by Doppler shifts, so we only measure *radial* components of galaxy velocities (the relativistic transverse Doppler effect has never been seen in a gravitational dynamics situation!).

So, suppose we measure the radial velocity, v_z, for many (100 or so) galaxies in a cluster. Then we get a distribution of velocities, and we can find the mean square dispersion of galaxy velocities about the *systemic radial velocity*, v_s. The systemic velocity is the mean velocity of the system,

$$v_s = \frac{1}{N_{GC}} \sum v_z \quad ,$$

where the sum is over cluster members only. The radial velocity dispersion, σ_z, is given by

$$\sigma_z^2 = \langle (v_z - v_s)^2 \rangle = \frac{1}{N_{GC}} \sum (v_z - v_s)^2 \quad .$$

But we need the total (three-dimensional) velocity dispersion, $\sqrt{\overline{v^2}}$, in the virial theorem. This is

$$\begin{aligned} \overline{v^2} = \langle v^2 \rangle &= \langle (v_x - v_{sx})^2 + (v_y - v_{sy})^2 + (v_z - v_{sz})^2 \rangle \\ &= 3 \langle (v_z - v_s)^2 \rangle \\ &= 3\sigma_z^2 \quad . \end{aligned}$$

This result is true if the velocity distribution is isotropic (as we would expect for a spherical cluster). Therefore, we can use the *observable* radial velocity dispersion, σ_z, to infer the *unobservable* three-dimensional velocity dispersion, $\sqrt{\overline{v^2}}$. In terms of σ_z, the virial theorem becomes

$$\boxed{GM = 3 R_e \sigma_z^2 \quad .}$$

Note the importance of the assumption of spherical symmetry, which provides $\overline{v^2} = 3\sigma_z^2$. This assumption allows us, once again, to solve the ubiquitous problem of the hidden dimensions ... in this case, also the two hidden velocities v_x and v_y.

b) *Measurement of R_e, the effective gravitational radius*

R_e, the effective gravitational radius, is defined by

$$\frac{1}{R_e} = \frac{1}{M^2} \sum_{\text{pairs}} \frac{m_i m_j}{r_{ij}} \quad ,$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass of a group of N point masses with separations $\{r_{ij}\}$. If all the m_i are the same, say m, then

$$\frac{1}{R_e} = \frac{1}{N^2} \sum_{\text{pairs}} \frac{1}{r_{ij}} \quad .$$

How can we estimate R_e from observations of a cluster of masses in which we observe only the projected separations between objects, p_{ij}, not the true (3-D) separations r_{ij}?

$\sum \frac{1}{p_{ij}}$ cannot be used as an estimator of $\sum \frac{1}{r_{ij}}$ because the few pairs with $p_{ij} \sim 0$ dominate the sum, giving a very bad estimate for R_e (since these small p_{ij}

are the hardest to measure accurately), although mathematically $\sum \frac{1}{p_{ij}}$ and $\sum \frac{1}{r_{ij}}$ are closely related if the vectors $\mathbf{r}_{ij}$ are randomly aligned.

Instead, it is better to assume spherical symmetry and then to use the properties of the observed projected distribution to get $\left\langle \frac{1}{r_{ij}} \right\rangle$. Suppose that we measure a number of galaxies/unit area on the plane of the sky $\nu(p)$, where p is the distance from the center of the cluster. Then $2\pi\nu(p)p\,dp$ is the number of galaxies in the annulus between projected radii p and $(p + dp)$.

The surface density profile, $\nu(p)$, is the projection of the true 3-D density distribution, $n(r)$, the number of galaxies/unit volume in the cluster.

$$\nu(p) = \int_{-\infty}^{\infty} n(r)\ dz \quad ,$$

where the integral is along the line of sight, z, and

$$r^2 = p^2 + z^2 \quad .$$

The problem is then to calculate $n(r)$ from the observed $\nu(p)$. This "inversion problem" is difficult and is another example of the *ubiquitous problem of the 3rd dimension* in astrophysics.

The standard method for solving this problem is *Plummer's method of strip counts*. Suppose that we count the number of galaxies in a one-dimensional strip between x and $(x + dx)$ from the center of the cluster. Let this number be $S(x)dx$. Then $S(x)$ is a projection of $\nu(p)$ in the y direction:

$$S(x) = \int_{-\infty}^{\infty} \nu(p)\ dy \quad ,$$

where

$$p^2 = x^2 + y^2$$

(note that I've changed from cylindrical (p, ϕ, z) coordinates to Cartesian (x, y, z) coordinates).

Using the expression for $\nu(p)$,

$$S(x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\ n(r) \quad .$$

Change to polar coordinates in the y-z plane:

$$\varpi^2 = y^2 + z^2$$

so that

$$\int\int dy\ dz \implies \int 2\pi\,\varpi\,d\varpi$$

and

$$r^2 = \varpi^2 + x^2$$

so that

$$\int \varpi\,d\varpi \implies \int r\ dr \quad .$$

Then

$$S(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy\ dz\ n(r) = \int_{0}^{\infty} 2\pi\,\varpi\,d\varpi\ n(r) = \int_{x}^{\infty} 2\pi\ r\ dr\ n(r)$$

and so

$$\frac{dS}{dx} = -2\pi\ x\ n(x) \quad ,$$

i.e.,

$$\boxed{n(r) = -\frac{1}{2\pi r}\left(\frac{dS(x)}{dx}\right)_{x=r} \quad .}$$

So, by doing the 1-D strip counts, we can get $n(r)$; but the appearance of a differential operator in this result makes it difficult to obtain an accurate $n(r)$ from noisy measurements of $S(x)$. How can we use this result to derive R_e? $M(r)$, the mass enclosed within radius r, is

$$M(r) = \int_{0}^{r} 4\pi r^2 dr\ m\ n(r) \quad ,$$

and the potential energy is

$$U = -\int_{0}^{\infty} \frac{GM(r)\,dM(r)}{r} \quad ,$$

so that, from the *definition*

$$R_e = -\frac{G[M(R)]^2}{U} \quad ,$$

R_e can be deduced. [$r = R$ is the edge of the mass distribution: $n(r) = 0$ for $r > R$.]

In fact, we can find a simple formula for R_e:

$$
\begin{aligned}
M(r) &= -\int_0^r 4\pi x^2 dx\ m\ \frac{1}{2\pi x}\cdot\frac{dS}{dx} \\
&= -2m\int_0^r x\ \frac{dS}{dx}\ dx \\
&= -2m\,[xS(x)]_0^r + 2m\int_0^r S(x)\ dx \\
&= 2m\left\{\int_0^r S(x)\ dx - r\,S(r)\right\}
\end{aligned}
$$

so that the total mass, $M(R)$, is simply

$$
M(R) = 2m\int_0^R S(x)\ dx \quad ,
$$

since $S(R) = 0$. Similarly, for the gravitational potential,

$$
\begin{aligned}
U &= -\int_0^R \frac{G}{r}\left\{\int_0^r 4\pi\,r^2\,m\,n(r')\,dr'\right\} 4\pi\,r^2\,dr\,m\,n(r) \\
&= -16\pi^2\,Gm^2\int_0^R n(r)\,r\,dr\int_0^r n(r')\,r'^2\,dr' \quad .
\end{aligned}
$$

But

$$
n(r) = -\frac{1}{2\pi r}\ \frac{dS}{dr} \quad ,
$$

so that

$$
\begin{aligned}
U &= -4Gm^2\int_0^R \frac{dS}{dr}\ dr\ \int_0^r\ r'dr'\ \frac{dS(r')}{dr'} \\
&= -4Gm^2\int_0^R \frac{dS}{dr}\ dr\ \left\{[r'S(r')]_0^r - \int_0^r\ S(r')\,dr'\ \right\} \\
&= -4Gm^2\int_0^R \frac{dS}{dr}\ dr\ \left\{rS(r) - \int_0^r S(r')\,dr'\right\}
\end{aligned}
$$

$$= -4Gm^2 \left\{ \left[r \frac{1}{2} S^2(r) \right]_0^R - \int_0^R \frac{1}{2} S^2(r) \, dr \right\}$$

$$+ 4Gm^2 \left\{ \left[S(r) \int_0^r S(r') \, dr' \right]_0^R - \int_0^R S(r)^2 \, dr \right\}$$

$$= 2Gm^2 \int_0^R S^2(r) \, dr - 4Gm^2 \int_0^R S^2(r) \, dr$$

$$= -2Gm^2 \int_0^R S^2(r) \, dr \quad .$$

Then, using it is easy to show that

$$\boxed{R_e = 2 \frac{\left[\int_0^R S(x) dx \right]^2}{\left[\int_0^R S^2(x) dx \right]} \quad .}$$

This is an attractive result: it provides a *direct* route to R_e from plate material. Since it involves only integrals of observed quantities, it should be accurate.