Mechanics

Dr Andrew Young

2025-11-10

Table of contents

Preface

This text covers the Classical Mechanics content of the first year core physics unit at the University of Bristol. It is divided into three parts, the Course notes, Problem sheets, and Workshops.

I would like to thank Dr Simon Hanna who taught this course previously, and the course textbooks, on which this course is based.

There are certainly errors in the text that are mine. Please email Andy.Young@bristol.ac.uk with any corrections.

This documents was written using Quarto. The notes are available in HTML5 (the colour scheme and text size can be changed, and is good for screen readers), PDF, and Word format. This improves accessibility somewhat, but I am happy to receive feedback where you have run into problems (e.g., with equations). To learn more about Quarto books visit https://quarto.org/docs/books.

Part I

Course notes

1 Introduction

Welcome to the first year "Core Physics" unit at the University of Bristol. This unit is divided into four sections, covering Classical Mechanics, the Properties of Matter, Oscillations and Waves, and Quantum Mechanics. These notes are for the Mechanics section of the course.

1.1 Why study mechanics?

In addition to learning about mechanics, we will learn how to simplify problems, translate them into mathematical language, and explore the solutions. This is a skill that will be useful in many other areas of physics and beyond. Examples of mechanics problems we'll be able to think about include the following.

- How do you make a space telescope point in the right direction?
- What impulse do I need to give a defective object to knock it off a production line, and where will it end up?
- I'd like to store energy when demand for electricity is low. Maybe energy could be stored in a flywheel, or by lifting massive weights. Are these schemes feasible and which would be best?
- What torque do the electric motors in my car need to provide if I want to accelerate from 0 to 60 mph in under 3 seconds?

A good knowledge of classical mechanics will provide a foundation for future courses. Often more advanced problems reduce to something that looks like a classical mechanics problem.

1.2 Course logistics

You should familiarise yourself with the Blackboard page for the unit. This is the primary source for information about the unit. The lecture, problems class, and workshop schedule should appear in your personal timetable.

There will be 15 Classical Mechanics lectures delivered over 5 weeks. While you have these notes, it will be useful for you to make some notes yourself. In addition, it really helps to learn physics by *doing* physics. To support the development of your problem solving skills there will be problems classes, and workshops. The sixth week will be a "consolidation" week where the are no new mechanics lectures so you can review or catch up on the course so far.

You should check your personal timetable to confirm the times and locations of the lectures, workshops, and problems classes.

1.3 Textbooks

There are several textbooks relevant to mechanics and the core physics content in the first year. These books are available in the library and on-line as eBooks. It is good to get in the habit of reading around the subject and looking at textbooks. These will provide a different viewpoint of the same material which you might find helps your learning, they also contain examples you might want to look at, as well as additional material that is not covered in the lectures. While this additional material is not examinable, it might be useful to you in the future.

- Tipler, Mosca, and Mosca (2008) is a general textbook that covers all of first year core physics. Classical Mechanics is a well established field and we will generally follow the structure and content as this book. This is available as an eBook which you can find by following the "Reading list" link in Blackboard (it will be on the panel on the left).
- Kleppner and Kolenkow (2014) is a more specialised textbook that covers mechanics at an introductory level.
- Goldstein et al. (2014) is an advanced textbook that covers all of the classical mechanics you are likely to need at university level.

1.4 Why should you come to lectures?

You are part of a community of like-minded students all interested in studying Physics. You can discuss the topics covered in lectures, and ask each other questions as you learn together. It is important to remember that your degree is not a competition. There are no "quotas" and everyone can do well. You can help each other!

What if you find lectures boring? You will still have something in common to talk about with your friends. If you are daydreaming you can think about how some of the concepts can be generalised, or taken to the extreme. For example, we might be working on a one-dimensional problem, but could that be generalised to 2, 3, 4 or even infinite-dimensions? What happens in the limit as things get faster or smaller – you might end up in a regime where relativity or quantum mechanics are important. Even the simplest things can be quite profound if you think about them deeply (Einstein was good at this). If you are daydreaming in lectures maybe try to daydream about physics!

Finally, I hope to convey some of my enthusiasm for physics to you, and explain *why* we're covering this content in mechanics. I hope you will find the lectures interesting and enjoyable. I am always happy to discuss physics and answer questions after lectures.

2 Motion in 1D

Displacement, Velocity, Acceleration

2.1 Units, dimensions, estimation

Before we start to think about mechanics it will be really useful to discuss units, dimensions, and estimation. This is important because all of the physics equations you write down must be dimensionally correct otherwise they won't make sense (e.g., "how long is piece of string?", answer "15 seconds" is not sensible). Estimation will also be extremely useful for checking your answers (e.g., "how long will it take to warm up my soup?", answer "5×10^{17} seconds" is not sensible; a universe with intelligent life can be created in less time).

2.1.1 Dimensional analysis

Each term in an equation has to have the same "dimensions", i.e., *units*, for the equation to make sense. Units can be combined by multiplication and division. Area, A for example, is given by multiplying two lengths together so has dimensions of length squared. We use square brackets to denote "dimension of", so this is written $[A] = L^2$ to show that area has has the dimensions of length squared. The units of some common quantities are given in Table 2.1. You can add to this list as you encounter new quantities and units.

Table 2.1: Dimensions of some physical quantities. There are many others that you can calculate yourself.

Quantity	Dimension
Length	L
Mass (M)	M
Time	T
Area (A)	L^2
Volume (V)	L^3
Speed	L/T

Quantity	Dimension
Acceleration	L/T^2
Force (F)	ML/T^2
Pressure (F/A)	M/LT^2
Density (M/V)	M/L^3
Energy	ML^2T^{-2}
Power	ML^2T^{-3}

The superpower you gain with dimensional analysis is the ability to generate plausible equations. But with great power comes great responsibility; the equations you generate this way don't *have* to be correct but they often provide useful insight. Let's look at an example.

Worked example

A mass on the end of a string is moving in a circle at constant speed. What is the force exerted on the mass by the string?

Answer

What might the answer depend on? Maybe the mass m, speed v, radius r of the circle. These have dimensions

$$[m] = M$$

$$[v] = L/T$$

$$[r] = L$$

and the dimension of the force, F, is

$$[F] = ML/T^2$$

So we need to combine m, v, and r to get something that has the same dimensions as F. We find that the following combination works

$$F \propto \frac{mv^2}{r}.$$

We've used "proportional to", $\propto$, because we can always multiply by a constant that has no dimensions. In this case, the constant is 1, and we know this is the right answer!

2.1.2 Significant figures

A friend asks you, "How old is the universe?" Conveniently, another friend, who is an astronomer, had just told you that the Universe is 14 billion years old. So you answer, "The universe is 14 billion years and 2 hours old."

In physics it is important to use an appropriate number of *significant figures* to give some idea of the precision of a measurement. A reliably known digit is called a significant figure. So, e.g., if we measure the height of a person to be 1.7 m, we are saying that we know the length to within 0.1 m. If we measure the length of a table to be 1.75 m, we are saying that we know the length to within 0.01 m. The number of significant figures is the number of digits that are known with certainty plus one estimated digit. So, e.g., if we measure the length of a table to be 1.753 m, we are saying that we know the length to within 0.001 m.

Significant figures

- Significant figures are "reliable digits" plus one "estimated digit".
- When multiplying or dividing quantities, the result should have the same number of significant figures as the quantity with the fewest significant figures.
- When adding or subtracting quantities, the result should have the same number of decimal places as the quantity with the fewest decimal places.

2.1.3 Estimation

Estimation works by using "orders of magnitude". For example, the height of a person is $\sim 10^0 = 1$ m. Even though the average height of a person is closer to 2 m and there are variations between people, this doesn't matter if we're interested in making an order of magnitude estimate. Why might this be helpful? It immediately tells us that if, e.g., we want to build a new lecture theatre we should be thinking about rooms that are $\sim 10^0$ m high (perhaps in the range 0.1–100 m), and certainly not $\sim 10^{-8}$ m or $\sim 10^{10}$ m. Similarly, if we performed some calculation to estimate the height of a person and the answer wasn't $\sim 10^0$ m we would know that we had made a mistake.

The power of these "order of magnitude" estimates is that we don't need particularly precise data. In the example above we only needed to know that the height of a person is roughly 10^0 m, and any reasonable guess would be good enough to rule out building an absurdly small or large building. Of course, being *more* precise would require more work.

Worked example

How many grains of sand are there on a patch of beach 500 m long and 100 m wide?

Answer

This sounds impossible to answer, and it is practically impossible if you want to know *precisely* the number of grains. However, an estimate is quite feasible.
We need to guess the size of a grain of sand, and the depth of the sand on a beach. Let's say a grain of sand has a diameter of 1 mm, and the depth of sand is 2 m.
The volume of sand on the beach is approximately

$$V_\mathrm{b} \simeq 500 \times 100 \times 2 = 10^5 \ \mathrm{m}^3.$$

The volume of a spherical grain of sand is approximately

$$V_\mathrm{s} \simeq \frac{4}{3}\pi \left(\frac{1.0 \times 10^{-3}}{2}\right)^3 \simeq 10^{-9} \ \mathrm{m}^3$$

(you can approximate π as 3 to do these calculations in your head).
The number of grains is then

$$N \simeq \frac{V_\mathrm{b}}{V_\mathrm{s}} \simeq 10^{14}.$$

where we have neglected the packing efficiency of grains (there is some space between grains), and assumed the grains fill up most of the volume.
Note that this estimate doesn't change much if, e.g., the depth is 1 m or 3 m, or if the diameter of a grain of sand is 1.5 mm or 2 mm, as long as we have a reasonable, *order of magnitude* estimate.

Remember that you can use dimensional analysis and estimation throughout the course.

2.1.4 Some examples

It is always good to consolidate your physics knowledge by trying some example questions.

Question to try at home – dimensional analysis

The drag force on a sphere falling through the air depends on the air density, ρ, the radius of the sphere, r, and the speed of the sphere, v. Use dimensional analysis to come up with an equation for the drag force.

2.2 Displacement, velocity, and acceleration

OK, on to mechanics!

We are interested in determining the *equations of motion* of physical systems that describe how they evolve as a function of time. For example, how far a train has moved along a track, how fast is it moving, what is its acceleration? To do this we will need to know something about vectors and calculus. These concepts will become increasingly useful throughout your physics degree.

Let's start by looking at motion in one dimension to familiarise ourselves with the basics of displacement, velocity, and acceleration. We'll then generalise things to two and three dimensions, making use of vectors. (And ultimately to four dimensions, but that's a story for another time).

2.2.1 Displacement

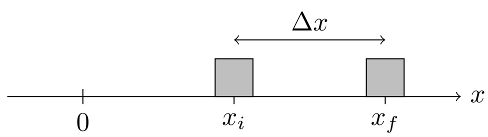

Figure 2.1: The one dimensional displacement is the difference between the final and initial position, $\Delta x = x_f - x_i$.

Consider a train moving along a track. At some initial time t_i the train is a distance x_i along the track. At some later time t_f the train has moved further along the track to a position x_f. The *displacement* of the train is the difference between the final and initial positions,

$$\Delta x = x_\mathrm{f} - x_\mathrm{i}.$$

The Greek capital letter Δ, "delta", is often used to denote a large change such as this. (A lower case delta is often used to represent small changes, $\delta \ll 1$).)

2.2.2 Average velocity

The *velocity* is the rate at which position changes. The average velocity, v_av is given by the change in position divided by the change in time,

$$v_\mathrm{av} = \frac{\Delta x}{\Delta t}.$$

where $\Delta t = t_\mathrm{f} - t_\mathrm{i}$ is the change in time.

This means that the

$$\text{average velocity} = \frac{\text{total displacement}}{\text{total time}}.$$

which is the gradient of a straight line from $(x_\mathrm{i}, t_\mathrm{i})$ to $(x_\mathrm{f}, t_\mathrm{f})$.

Worked example

It has been speculated that the Oumuamua comet that passed through the solar system in 2017 was an alien spacecraft. It was observed leaving the solar system, moving along an approximately straight line. On 7$^{\text{th}}$ October it was observed to be 1.47×10^{11} m from the Sun and on 28$^{\text{th}}$ October is was observed to be 2.29×10^{11} m from the Sun. What was its average velocity?

Answer

The average velocity is given by

$$v_{\text{av}} = \frac{\Delta x}{\Delta t}$$

In this case $\Delta x = (2.29 - 1.47) \times 10^{11} = 8.2 \times 10^{10}$ m and $\Delta t = 21$ days $= 1.8 \times 10^6$ s. So the average velocity is

$$v_{\text{av}} = \frac{8.2 \times 10^{10}}{1.8 \times 10^6} \text{ m/s} = 4.6 \times 10^4 \text{ m/s} = 46 \text{ km s}^{-1}$$

This is larger than the escape velocity of the solar system at this distance ($\simeq 17$ km s^{-1}). When we discuss energy later in the course you'll be able to figure this out yourselves.

2.2.3 Relative velocity

When we are talking about velocity, we are considering a specific *frame of reference* that our coordinate system is attached to.

Frame of reference

A frame of reference is an extended object whose parts are at rest relative to each other.

The velocity will be different in different frames of reference. For example, consider walking along the aisle of a train. If you are walking towards the front of a train at $v_{\text{t}} = 1$ m s^{-1}, and the train is moving at $v_{\text{tp}} = 10$ m s^{-1} relative to the platform, then you are moving at $v_{\text{p}} = 11$ m s^{-1} relative to the platform. This is called the *relative velocity.*

$$v_{\text{p}} = v_{\text{t}} + v_{\text{tp}}.$$

Interesting note

This equation is *not true* in general and only applies for small velocities $v \ll c$, where c is the speed of light. For velocities that are a significant fraction of the speed of light we have to consider *special relativity*. You will study this next year, or you can look it up in the course textbook.

2.2.4 Instantaneous velocity

In general, the velocity can change over time. For example, the train might be speeding up or slowing down.

How can we define the velocity at an *instant* in time? We can look at the average velocity over progressively smaller intervals of time. This is shown in Figure 2.2, where we are considering the velocity at some time t_0 and position x_0, denoted by the point P in the figure. As we estimate the velocity using shorter and shorter time intervals, our velocity estimate gets closer and closer to the *tangent* at point P.

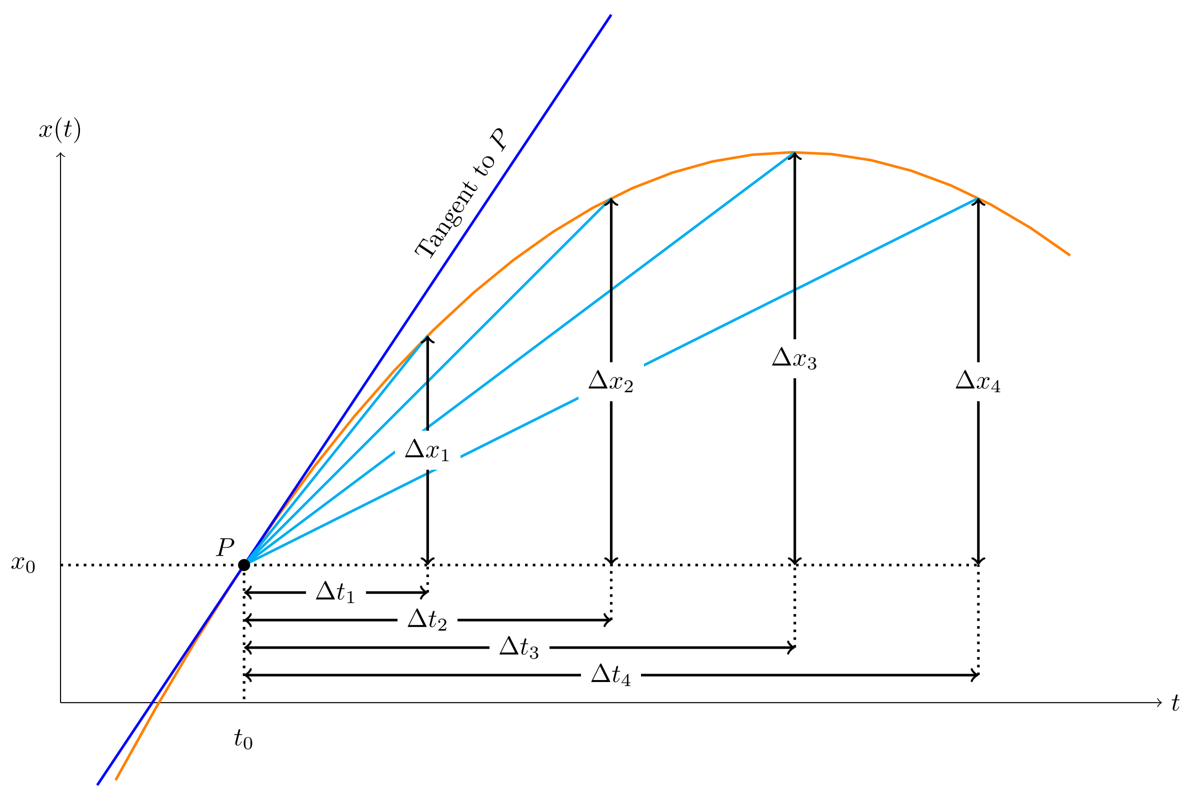

Figure 2.2: Instantaneous velocity.

We write this mathematically at a *limit*,

$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

which is the slope of the curve at point t_0. This is the derivative of position x with respect to time t, and speed is the rate of change of position.

$$v = \frac{dx}{dt}$$

This is often written as $v = \dot{x}$, and we say "x dot".

> **Definition of derivative**
>
> The derivative of a function $f(t)$ with respect to t is defined as
>
> $$\frac{df(t)}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$
>
> You can try this with, e.g., $f(t) = t^2$.

> **Example**
>
> Galileo drops a ball from the top of the leaning tower of Pisa. The position of the ball is described approximately as $s(t) = 5t^2$ m where the time, t, is measured in seconds after the ball is released, and the vertical position, x, is measured downwards in metres. What is the velocity at any time t?

> **Answer**
>
> The velocity is the derivative of position with respect to time,
>
> $$v = \frac{ds}{dt} = 10t \text{ m s}^{-1}.$$

2.2.5 Acceleration

Similarly, the acceleration, a, is given by the rate of change of velocity with time,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

which can also be written

$$a = \ddot{x}$$

and we say "x double dot".

It is clear that some knowledge of calculus is going ot be useful for solving mechanics problems. In fact, calculus is going to be *really* important throughout your physics degree. It is therefore highly recommended that you practice differentiating some standard functions so that you can concentrate on the underlying physics rather than the mathematics.

i Differentiation examples

Here are some examples of functions that you should be able to differentiate by the end of the course. You will have more practice of this elsewhere in your degree programme.

- Polynomial functions such as $y(x) = ax^n + bx^m$.
- Trigonometric functions such as $y(x) = \sin(x)$.
- Natural logarithms such as $y(x) = \ln(x)$.
- Product rule for $y(x) = f(x)g(x)$.
    - $y(x) = x^2 \sin(x)$.
- Chain rule for $y(x) = f(g(x))$.
    - $y(x) = \sin(x^2)$.

2.2.6 Integration

So far we have used differentiation to get the "rate of change" of certain quantities. Velocity is the rate of change of position; acceleration is the rate of change of velocity. How do we go the other way?

We use integration, which is the opposite of differentiation. The *fundamental theorem of calculus* which states that

$$f(x) = \int_0^x \frac{df(x')}{dx'} dx'.$$

Note the use of a *dummmy variable*, x', in the integral. This is necessary because the variable x is already being used as the upper limit of the integral and we want the dummy variable x' to vary from 0 to x.

For example, if we have a graph of velocity versus time and we want to know the change in displacement we can integrate.

$$\int_{x_0}^{x_1} dx = \int_{t_0}^{t_1} v dt$$

which is the area under the curve between times t_0 and t_1. This gives us

$$\Delta x = x_1 - x_0 = \int_{t_0}^{t_1} v dt$$

It is easy to see why this is the case for constant velocity. However, if you imagine dividing the timeline into segments where the velocity is approximately constant you visualise how the general result holds.

> **Integration examples**
>
> Here are some examples of functions that you should be able to integrate by the end of the course. You will have more practice of this elsewhere in your degree programme.
>
> - Polynomial functions such as $y(x) = ax^n + bx^m$.
> - Trigonometric functions such as $y(x) = \sin(x)$.
> - Integration to obtain natural logarithms such as $y(x) = 1/x$.
> - Integration by parts.
> - Integration by substitution.

2.2.7 Constant acceleration

It is very useful to consider a system in which there is constant acceleration, a. We can write this acceleration in the following way.

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = a$$

We can integrate this equation,

$$\int_{v_0}^{v_1} dv = \int_{t_0}^{t_1} a dt$$

$$v_1 - v_0 = a(t_1 - t_0)$$

We can start measuring time from $t_0 = 0$ to get

$$v_1(t_1) = v_0 + at_1$$

where t_1 is the elapsed time. Note that v_1 is a function of t_1 because it depends on t_1. Different elapsed times result in different velocities.

Integrating once more, again assuming we start timing at $t_0 = 0$, we get

$$x_1 - x_0 = \int_0^{t_1} v(t) dt$$

$$x_1 - x_0 = \int_0^{t_1} (v_0 + at)dt$$

If we measure displacement from $x_0 = 0$ we get

$$x_1(t) = v_0 t_1 + \frac{1}{2}at_1^2$$

Finally, we can derive one more equation using an integration "trick". Note that we can write

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

where we have used the chain rule, considering v to be a function of distance, $v(x)$. We can then write constant acceleration as

$$v\frac{dv}{dx} = a$$

and rearrange to get

$$\int_{v_0}^{v_1} v dv = \int_{s_0}^{s_1} a dx$$

$$\frac{v_1^2}{2} - \frac{v_0^2}{2} = a(s_1 - s_0)$$

Again assuming $s_0 = 0$ we get

$$v_1^2 = v_0^2 + 2as_1$$

These are the equations of motion for constant acceleration. They are more often written as follows.

2.2.8 SUVAT equations

SUVAT equations

You will often hear reference to the "SUVAT" equations. These are the set of equations derived above that describe motion with constant acceleration. This term is frequently used in the UK A-Level education system, but you might not hear it elsewhere. The equations are:

- s is displacement
- u is the initial velocity
- v is the final velocity
- a is the constant acceleration

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

It is worth remembering these, as well as knowing how to derive them from first principles.

2.2.9 Variable acceleration

In general, the acceleration will not be constant. It this is the case, we cannot use the "SUVAT" equations. Instead, we have to write down the equations of motion and solve them using calculus.

3 Vectors in 2D and 3D

So far we've been looking at motion in one dimension. However, in general we are interested in motion in two and three dimensions. To describe motion in two and three dimensions it is very convenient to use vectors.

Vectors will be used extensively throughout your physics degree. Later on you will find that vectors can be generalised in all sorts of interesting ways so it is important to have a solid understanding of the fundamentals, even if they can seem deceptively straightforward at first. You'll find that surprisingly deep insights can often come from seemingly simple ideas!

In mechanics we will use vectors to describe things like the position of an object, its speed and direction of motion (velocity), and acceleration.

3.1 Vector properties and algebra

3.1.1 Vector addition

You can think of vectors as lines with a particular length and direction (see Figure 3.1). These could be *displacement vectors* representing the distance and direction between two points in space.

We will write vectors with an arrow above them, e.g., $\vec{a}$, but you might also see vectors written in bold, $\mathbf{a}$, or underlined, $\underline{a}$. It doesn't matter what notation you use as long as you remain consistent and are clear about which quantities are vectors, and which are regular numbers or scalars.

> Definition of a vector
>
> Vectors are quantities that have both a magnitude and a direction that can be added together like displacements (see Figure 3.1).

There are several important properties that vectors have that you should be familiar with.

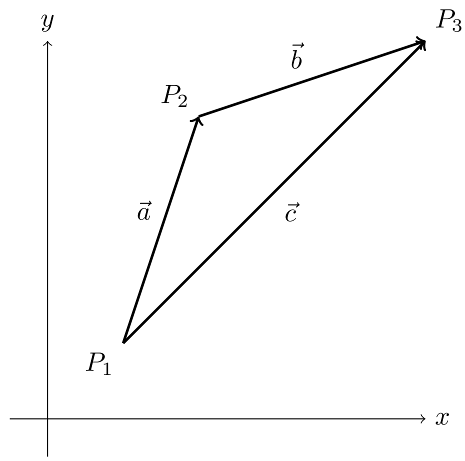

Figure 3.1: A particle moves from point P_1 to point P_2, with a displacement given by the vector $\vec{a}$. The particle then moves to point P_3, with a displacement given by the vector $\vec{b}$. The overall resultant displacement from P_1 to P_3 is given by the vector $\vec{c} = \vec{a} + \vec{b}$.

3.1.2 Magnitude

The *magnitude* of a vector is its length. The magnitude of the vector $\vec{a}$ is a scalar, a, which is written as

$$a = |\vec{a}| \, .$$

3.1.3 Multiplication by a scaler

Multiplying a vector by a scalar produces another vector that is parallel to the original vector, but with a different magnitude.

$$\vec{b} = \lambda \vec{a}$$

where λ is a scalar. The magnitude of this new vector is

$$b = \left|\vec{b}\right| = |\lambda \vec{a}| = |\lambda| \, |\vec{a}| = |\lambda| \, a.$$

The direction of the new vector is the same as the original vector (i.e., parallel) if $\lambda > 0$, and opposite (i.e., anti-parallel) if $\lambda < 0$.

3.1.4 Unit vector

A vector that has length 1 is said to be a *unit vector*. We can convert a vector to a unit vector by dividing by its length (equivalent to multiplying by 1 / length). We denote a unit vector by writing a “hat” over the vector.

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|},$$

which has magnitude 1,

$$|\hat{a}| = 1.$$

3.1.5 Addition and subtraction of vectors

We have seen how to add vectors in Figure 2.1. The sum of two vectors is another vector. The sum of two vectors $\vec{a}$ and $\vec{b}$ is written as

$$\vec{c} = \vec{a} + \vec{b}.$$

Subtracting $\vec{b}$ from $\vec{a}$ can be achieved by *adding* $-1 \times \vec{b} = -\vec{b}$ to $\vec{a}$. This is illustrated in Figure 3.2.

$$\vec{c} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

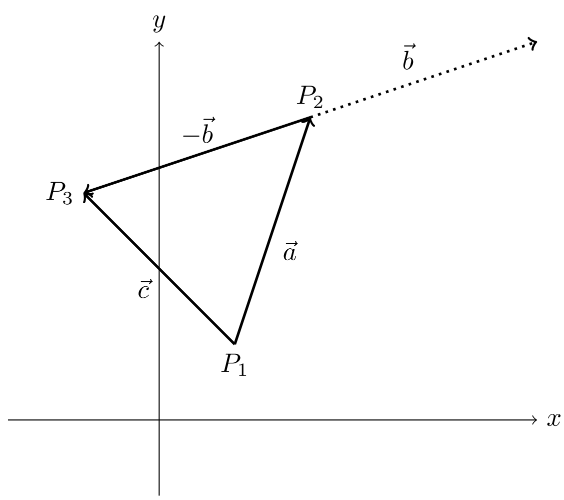

Figure 3.2: Vector subtraction. In this case $\vec{c} = \vec{a} - \vec{b}$.

3.1.6 Summary of vector properties

Summary of vector algebra

There are important algebraic rules that vectors obey. These are:

- Commutative law
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$
- Associative law
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$
- Distributive law
$$\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$$
where α is a scalar.

These might seem obvious for vectors, but you will encounter other objects later on that do not obey these rules. You can try to prove these, e.g., by sketching a diagram.

3.2 Components of a vector

So far we have considered vectors as directed line segments without specific reference to any coordinate system. Everything we have defined so far is independent of any coordinate system. This is one of the benefits of using vectors. However, we often want to calculate quantities in a specific frame of reference, with a particular coordinate system. To do this we can write the components of a vector in terms of a coordinate system.

As a concrete example we will look first at cartesian coordinates in two dimensions. We can get the x-component of a vector by projecting the vector onto the x-axis. Similarly, we can obtain the y-component by projecting the vector onto the y-axis. This is illustrated in Figure 3.3.

We can also see from the figure that the angle between the vector $\vec{a}$ and the x-axis is given by

$$\tan(\theta) = \frac{a_y}{a_x}$$

and the magnitude of $\vec{a}$ is given by

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}$$

using Pythagoras' theorem.

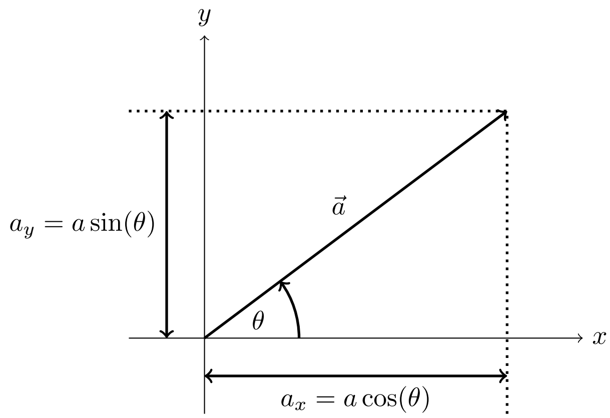

Figure 3.3: Components of a two dimensional vector a are obtained by projecting the vector onto the x-axis and y-axis.

3.2.1 Basis unit vectors

In cartesian coordinates in three dimensions we denote unit vectors in the x, y, and z directions as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, respectively. These are called basis vectors. Any vector can be written as a linear combination of scalars times these basis vectors. For example, the vector $\vec{a}$ can be written as

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}.$$

where a_x, a_y, and a_z are the *components* of the vector $\vec{a}$ in the x, y, and z directions, respectively. This is illustrated in Figure 3.4.

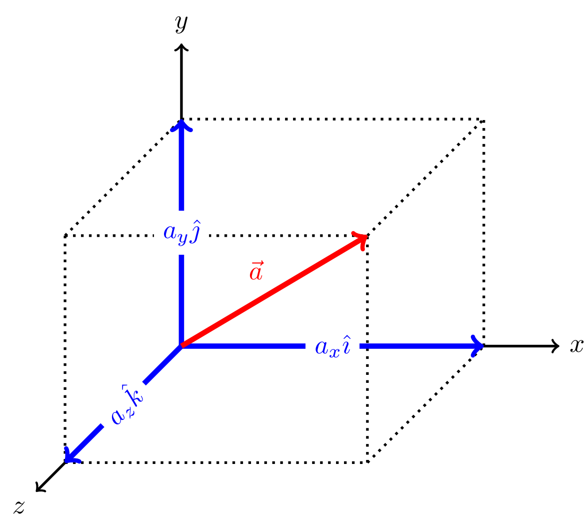

Figure 3.4: Components of a vector $\vec{a}$ in a three dimensional cartesian coordinate system.

We can also write the vector $\vec{a}$ as

$$\vec{a} = (a_x, a_y, a_z)$$

where the components of the vector are written in parentheses. This is called the *component form* of the vector.

One way to determine these components is using *direction cosines* by looking at the component of the vector projected onto the coordinate axes. The components of $\vec{a}$ are given by

$$a_x = |\vec{a}| \cos(\alpha)$$

$$a_y = |\vec{a}| \cos(\beta)$$

$$a_z = |\vec{a}| \cos(\gamma)$$

where α, β, and γ are the angles between $\vec{a}$ and the x, y, and z axes, respectively.

3.3 Vector position, velocity, and acceleration

The *position vector* of a particle is its vector displacement from the origin. The position vector of a particle at position (x, y, z) is given by

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}.$$

A change in position for a particle moving from position vector $\vec{r}_0$ at time t_0 to $\vec{r}_1$ at time t_1 is given by

$$\Delta\vec{r} = \vec{r}_1 - \vec{r}_0.$$

The *average velocity* vector is then

$$\vec{v}_{\text{av}} = \frac{\Delta\vec{r}}{\Delta t}$$

where $\Delta t = t_1 - t_0$. This closely follows our discussion of velocity in one dimension.

The instantaneous velocity is then given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}.$$

3.4 Relative velocity

This is a generalisation fo the one dimensional case, where the scalars have been replaced by vectors.

$$\vec{v}_{\text{p}} = \vec{v}_{\text{t}} + \vec{v}_{\text{tp}}$$

Example

A pilot wants to land a plane on a runway heading due north. The speed of the plane relative to the air is 70 m s^{-1}. There is a cross-wind of 15 m s^{-1} blowing from the west to the east. What direction should the pilot point the plane to land on the runway? How fast is the plane moving relative to the ground?

Answer

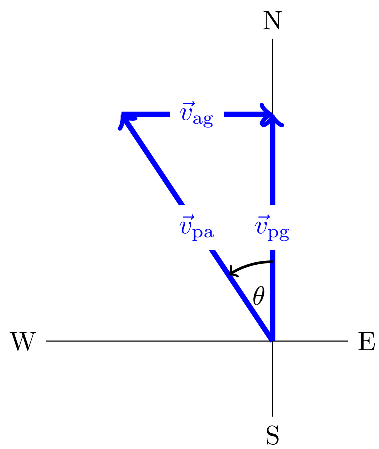

Figure 3.5: The desired direction is shown by the velocity of the plane relative to the ground, $\vec{v}_{\text{pg}}$. The velocity of the air relative to the ground is $\vec{v}_{\text{ag}}$, and the velocity of the plane relative to the air is $\vec{v}_{\text{pa}}$.

The wind will tend to blow the plane off course to the east. To compensate for this the pilot has to have some component of their motion into the wind. This is illustrated in Figure 3.5. We can see that the velocity of the plane relative to the ground, $\vec{v}_{\text{pg}}$, is given by

$$\vec{v}_{\text{pg}} = \vec{v}_{\text{pa}} + \vec{v}_{\text{ag}}.$$

The desired angle is then given by

$$\sin(\theta) = \frac{|\vec{v}_{\text{ag}}|}{|\vec{v}_{\text{pa}}|} = \frac{v_{\text{ag}}}{v_{\text{pa}}} = \frac{15}{70} = 0.214.$$

$$\theta = 12.1^\circ.$$

We can use Pythagoras' theorem to calculate the speed of the plane relative to the ground,

$$v_{\text{pg}} = \sqrt{v_{\text{pa}}^2 - v_{\text{ag}}^2} = \sqrt{70^2 - 15^2} = 68.4 \text{ m s}^{-1}.$$

3.5 Summary

Position, velocity, and acceleration

Position

$$\vec{x}(t)$$

Velocity

$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \dot{\vec{x}}$$

Acceleration

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}}$$

Note: we will use the convention that "speed" is a scalar and "velocity" as a vector.

4 Newton's first law

As physicists we want to understand how the Universe works. We want to be able to quantitatively describe the motion of objects, and to explain and predict their behaviour. The foundation of classical mechanics is described by Newton's laws.

Newton's laws of motion describe how objects move in response to forces. They are the foundation of classical mechanics. They are well tested and found to be a good description of "every-day" physics.

There are three laws, and we will go through each in turn.

4.1 Newton's first law and inertial reference frames

Why do objects start to move, change direction, or speed? Galileo and Newton realised that a *force* is *required* to change velocity. (Note that velocity is a vector so this includes a change in direction even if the speed remains constant). Newton found that the acceleration of an object is proportional to the force acting on it, i.e.,

$$\vec{F} \propto \frac{d\vec{v}}{dt}$$

where the constant of proportionality is the mass of the object giving (spoiler alert) Newton's second law

$$\vec{F} = m\vec{a},$$

but more on this later…

Classical mechanics relates the motion of objects to the forces that act on them. These are described by Newton's three laws of motion, and we shall go through each in turn.

> **Newton's first law**
>
> An object at rest will remain at rest, and an object in motion will remain in motion at constant velocity, unless acted upon by an external force.

Inertial reference frame

If no net force acts on an object, a reference frame in which the acceleration of the object remains zero is an *inertial reference frame*.

4.1.1 Galilean transformation

We can transform between two inertial reference frames using a *Galilean transformation*. Let's consider a one dimensional example. Two frames, A, and B, are initially coincident. Frame B is moving relative to frame A with a velocity v in the $x-$direction. This means that at time t the origin of frame B is at $x = vt$ when measured in frame A.

We can write the position of an object P in frame B in two ways. In terms of the coordinate system in frame B, the object is at a position x'.

In terms of the coordinate system in frame A, the object is at position

$$x = vt + x'$$

which is illustrated in Figure 4.1.

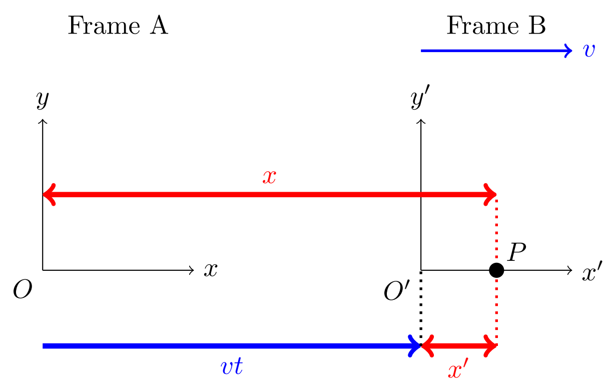

Figure 4.1: Galilean transformations between two inertial (i.e., not accelerating) frames A (with unprimed coordinates) and B (with primed coordinates).

The velocity of P measured in frame A is obtained by differentiation

$$\frac{dx}{dt} = v + \frac{dx'}{dt}$$

$$u = v + u'$$

where u is the velocity of P measured in frame A and u' is the velocity of P measured in frame B.

The acceleration can be obtained by differentiating again

$$\frac{d^2x}{dt^2} = 0 + \frac{d^2x'}{dt^2}$$

which is the **same** in both frames.

This also means that the forces are the same in both frames. The profound implication is that the laws of physics are the same in inertial frames! Furthermore, you can't tell the speed by measuring force.

Galilean transformations

Transformations between frame A (unprimed coordinates) and B (primed coordinates) are as follows.
Displacement x in A and x' in B.

$$x = vt + x'$$

Velocity u in A and u' in B.

$$u = v + u'$$

Acceleration a in A and a' in B are the **same**.

$$a = a'$$

These can be generalised to three dimensions using vectors.

Example

If an object is moving at a constant velocity in an inertial reference frame, which of the following can you deduce?

A) No forces act on the object
B) A constant force acts on the object in the direction of motion
C) The net force acting on the object is zero
D) The net force acting on the object is equal and opposite to its weight

Answer

C) If there is any net force, the object will accelerate.

Galilean relativity

Galilean relativity does not always work. For example, what happens when velocities approach the speed of light? You'll find out in the Special Relativity class next year (or you can read about it in the course textbook).

4.2 Conservation of momentum

Newton's first law tells us that *in the absence of external forces* a system's total momentum must be conserved. Momentum conservation is an expression of Newton's first law, i.e.,

$$\frac{d\vec{p}_{\mathrm{tot}}}{dt} = \vec{0}$$

where $\vec{p}_{\mathrm{tot}}$ is the total momentum of the system that is constant.

Consider a system made up of N particles with masses m_i, positions $\vec{r}_i$, and velocities $\vec{v}_i$. The total momentum of the system is

$$\vec{p}_{\mathrm{tot}} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \vec{r}_i\right) \tag{4.1}$$

where $\vec{p}_i$ is the momentum of the i^{th} particle.

Using the principle of moments (from A-level) the *Centre of Mass* (c.o.m.) of the system is

$$\sum_{i=1}^{N} m_i(\vec{r}_i - \vec{r}_{\mathrm{com}}) = \vec{0} \tag{4.2}$$

where $\vec{r}_{\mathrm{com}}$ is the position of the centre of mass. This means that the centre of mass is the point about which the moment of the system is zero. See Figure 4.2 for an illustration of the system.

We can write the total mass M as the sum of the individual masses

$$M = \sum_{i=1}^{N} m_i$$

We can rearrange Equation 4.2 to get an expression for the centre of mass

Centre of mass

$$\vec{r}_{\mathrm{com}} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i$$

Then from Equation 4.1 we see that the total momentum $\vec{p}_{\mathrm{tot}}$ is

$$\vec{p}_{\mathrm{tot}} = M\frac{d\vec{r}_{\mathrm{com}}}{dt} = M\vec{v}_{\mathrm{com}}$$

where $\vec{v}_{\mathrm{com}}$ is the velocity of the centre of mass.

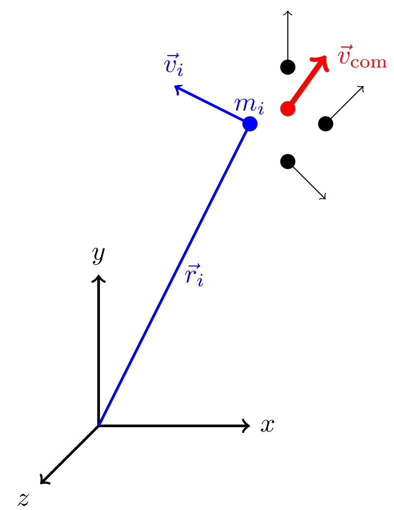

Figure 4.2: A system of N particles with masses m_i, positions $\vec{r}_i$, and velocities $\vec{v}_i$.

> **Momentum conservation**
>
> $$\text{Total momentum} = \text{total mass} \times \text{velocity of centre of mass}$$
>
> It doesn't matter what the individual particles are doing. We only need to think about the total mass and movement of the centre of mass.
> For a system of particles we can identify the centre of mass and its net movement.
> Under external forces the centre of mass behave like a point particle of mass M at $\vec{r}_{\text{com}}$.

4.3 Centre of mass frame

The "centre of mass frame", also known as the zero momentum frame, is an inertial frame moving at $\vec{v}_{\text{com}}$ so that in this frame $\vec{v}'_{\text{com}} = \vec{0}$. This is the zero momentum frame.

> **Example**
>
> An experiment of mass $2m$ is launched from the origin at an angle θ to the horizontal. In a previous identical test the projectile landed 55 m from the origin. However, in this test the experiment explodes at the highest point into into two pieces of equal mass. One piece has no horizontal motion, so falls directly downwards, landing at x_1, while the other continues to move horizontally, landing at x_2. This is illustrated in Figure 4.3. How far from the origin are x_1, and x_2? You should neglect air resistance.

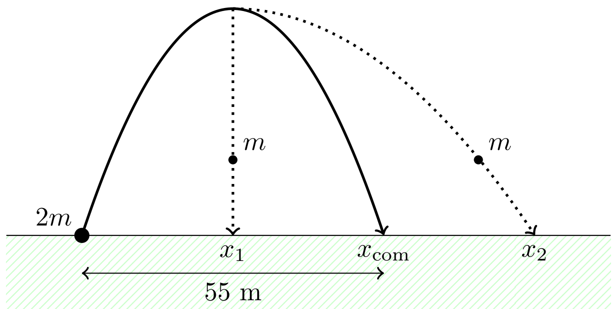

Figure 4.3: A projectile of mass $2m$ is launched from the origin. At its highest point it explodes into two pieces of equal mass. One piece falls vertically while the other continues to move horizontally.

Answer

Neglecting air resistance means that the trajectory of the centre of mass is symmetric about the highest point. This means that the highest point is at

$$x_1 = \frac{x_{\text{com}}}{2} = \frac{55}{2} = 27.5 \text{ m}$$

where x_{com} is the centre of mass range shown in Figure 4.3.
Momentum is conserved when the experiment explodes, so the centre of mass will follow the same trajectory as it did in the first test flight when the experiment did not explode. We know from the definition of centre of mass that

$$(2m)x_{\text{com}} = mx_1 + mx_2$$

$$2x_{\text{com}} = x_1 + x_2$$

We know x_{com} and x_1, so we can calculate

$$x_2 = 110 - 27.5 = 82.5 \text{ m}.$$

5 Newton's second law

Newton's first law tells us that an object at rest will remain at rest, and an object in motion will remain in motion at constant velocity, unless acted upon by an external force. We now look at what happens when a force is applied, introducing Newton's second law.

5.1 Newton's second law

The rate of change of momentum of a body is proportional to the applied force, $\vec{F}$, and takes place in the direction of the force,

$$\vec{a} = \frac{d\vec{v}}{dt} \propto \vec{F}$$

- This means that $\vec{a}$ is parallel to $\vec{F}$.
- The constant of proportionality is $1/m$.

In terms of momentum $\vec{p} = m\vec{v}$, this can be written as

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{5.1}$$

where $\vec{v}$ is the velocity of the body, m is its mass, and $\vec{a}$ is its acceleration.

In general $\vec{F}$ is the *net force*, i.e. the vector sum of all the forces acting on the body.

$$\vec{F} = \sum_i \vec{F}_i$$

If momentum is conserved

$$\frac{d\vec{p}}{dt} = \vec{0} \implies \vec{F} = \vec{0}$$

> Newton's second law
>
> The acceleration of an object is directly proportional to the net force acting on it, and the reciprocal of the mass of the object is the constant of proportionality.
>
> $$\vec{a} = \frac{\vec{F}_{\text{net}}}{m}$$

where

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_i$$

5.2 Impulse and momentum

In some problems, for example a collision between a golf club and a golf ball, a significant force is applied for a very short period of time. The force will vary during this time, increasing and then decreasing, although we might not know the precise way in which the force varies. In these case, it is useful to consider the *impulse* of the force, which is the integral of the force over the time for which it acts.

Impulse

The *impulse* $\Delta\vec{N}$ is defined as

$$\Delta\vec{N} = \int_{t_0}^{t_1} \vec{F} dt$$

i.e., the integrated force over some time interval. You can see that this is equivalent to the average force $\vec{F}_{\text{av}}$ times the time interval $\Delta t = t_1 - t_0$.

We can then use Newton's second law to get

$$\Delta\vec{N} = m\int_{t_0}^{t_1} \frac{d\vec{v}}{dt} dt = m\int_{\vec{v}_0}^{\vec{v}_1} d\vec{v}$$

where $\vec{v}_0$ and $\vec{v}_1$ are the initial and final velocities of the object at times t_0 and t_1, respectively.

Note that when we write $\int d\vec{v}$, we can write this as

$$\int d\vec{v} \equiv \int dv_x\hat{\imath} + dv_y\hat{\jmath} + dv_z\hat{k}.$$

$$\Delta\vec{N} = m(\vec{v}_1 - \vec{v}_0)$$
$$\Delta\vec{N} = \vec{p}_1 - \vec{p}_0$$

Therefore the impulse acting on a particle is equal to the change in momentum of the particle. This is called the "impulse-momentum" theory.

For a particle starting at rest

$$\vec{p}(t) = \int_0^t \vec{F}(t')dt'$$

Differentiating this we get that

$$\frac{d\vec{p}}{dt} = \vec{F}$$

i.e., force is rate of change of momentum.

Note: duration of impulse is often short and the shape of $\vec{F}(t)$ is generally *not* important, just its integral.

Example

A golf ball (mass 46 g) is struck by an average force of 2500 N for 1.0×10^{-3} s. Calculate the impulse transferred to the ball and the maximum velocity of the ball.

Answer

This question requires recognition that firstly the change in momentum, Δp, is equal to the impulse which acts on the body, and then that the impulse is "force × time".

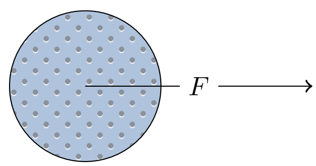

Figure 5.1: A golf ball being struck by a club with force F for a time $\Delta t = 1.0$ ms.

The impulse is given by Ft where Δt is the contact time between the golf club and ball.

$$\begin{aligned} F\Delta t &= 2500 \text{ N} \times 1.0 \times 10^{-3} \text{ s} \\ &= 2.5 \text{ N s} = 2.5 \text{ kg m s}^{-1} \end{aligned}$$

To calculate the veolcity we can use

$$F = m\frac{\Delta v}{\Delta t}$$

$$\Delta v = \frac{F\Delta t}{m} = \frac{2.5}{4.6 \times 10^{-2}} = 54 \text{ m s}^{-1}$$

This also brings up the opportunity to discuss dimensional analysis along with "appropriate precision"

a) Using dimensional analysis, the congruence of units between $F\Delta t$ and $m\Delta v$ can be seen
b) "Appropriate precision" can be demonstrated. The mass and the time are only reported to two significant figures, so the answers can only be reported to two significant figures.

5.3 Rocket equation

We can use our knowledge of Newton's laws, impulse, conservation of momentum, and inertial frames to derive an equation for the flight of a rocket. This is a bit tricky because the mass of the rocket changes with time as it burns through its fuel.

Before: Let's consider a rocket of mass m moving with velocity $\vec{v}$.

After: In a small time interval δt its mass changed by δm (which is negative because the mass decreases) and its velocity changes by an amount $\vec{\delta v}$.

This mass of burned fuel is ejected as exhaust out of the back of the rocket at a *relative velocity* $\vec{v}_{\text{rel}}$.

We need to conserve momentum for the rocket, fuel, and exhaust. The problem is illustrated in Figure 5.2 (in the rocket's frame) and Figure 5.3 (in our frame on the ground).

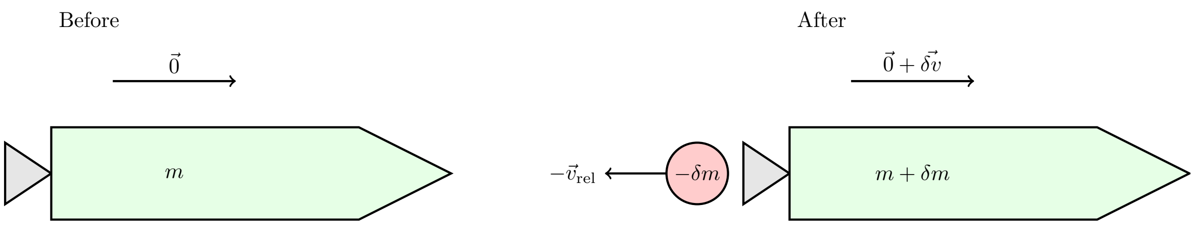

Figure 5.2: *Instantaneous rest frame of rocket:* A rocket of mass m changes its mass by δm (a negative change) by burning fuel. This means that a mass $-\delta m$ (which will be positive) of fuel emerges as exhaust at a velocity $\vec{v}_{\text{rel}}$. We're measuring positive displacements and velocities to the right so $\vec{v}_{\text{rel}}$ will be negative. In this frame the rocket is initially at rest and increases in velocity by $\vec{\delta v}$.

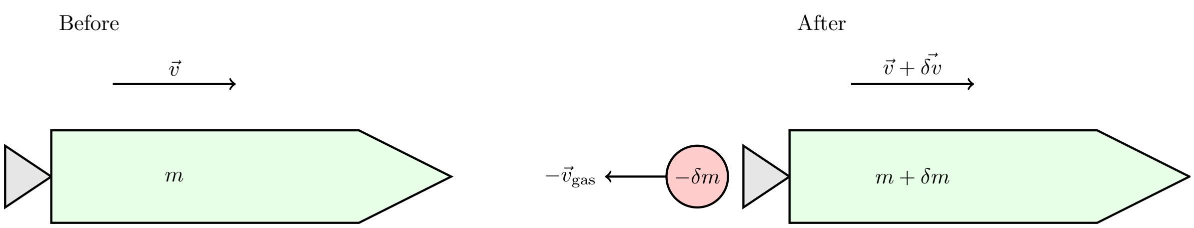

Figure 5.3: *Ground frame of reference:* A rocket of mass m changes its mass by δm (a negative change) by burning fuel. This means that a mass $-\delta m$ (which will be positive) of fuel emerges as exhaust at a velocity $\vec{v}_{\text{gas}} = \vec{v} + \vec{v}_{\text{rel}}$. We're measuring positive displacements and velocities to the right so $\vec{v}_{\text{gas}}$ will be negative as we are assuming $\vec{v}_{\text{rel}} > \vec{v}$.

We'll perform the analysis in the ground frame of reference because that is where we are measuring the rocket velocity. The initial momentum, which is conserved, is $m\vec{v}$. The velocity of the exhaust gas is $\vec{v}_{\text{gas}} = \vec{v} + \vec{v}_{\text{rel}}$, taking into account the relative velocity.

Conservation of momentum gives

$$m\vec{v} = (m + \delta m)(\vec{v} + \vec{\delta v}) + (-\delta m)(\vec{v} + \vec{v}_{\text{rel}})$$

Expanding and cancelling gives

$$\vec{0} = m\vec{\delta v} + \delta m\vec{\delta v} - \delta m\vec{v_{\text{rel}}}$$

Dividing this by δt we get

$$\vec{0} = m\frac{\delta\vec{v}}{\delta t} + \frac{\delta m\vec{\delta v}}{\delta t} - \frac{\delta m}{\delta t}\vec{v_{\text{rel}}}$$

Taking the limit at $\delta t \to 0$, we can ignore second order terms like $\delta m\delta\vec{v}$, to get

Rocket equation

$$m\frac{d\vec{v}}{dt} = \vec{v}_{\text{rel}}\frac{dm}{dt}$$

- $md\vec{v}/dt$ is the resultant thrust on the rocket,
- $\vec{v}_{\text{rel}}$ is in the opposite direction to the rocket, and
- $dm/dt < 0$,

so this makes sense.

Example

A rocket burns 400,000 kg of fuel in 200 seconds, producing an average thrust of 8×10^6 N. What is the speed of the exhaust?

Answer

$$\frac{dm}{dt} = \frac{400000}{200} = 2000 \text{ kg s}^{-1}$$

The thrust is given by the rocket equation above

$$8.6 \times 10^6 = v_{\text{rel}}\frac{dm}{dt} = v_{\text{rel}} \times 2000$$

$$\implies v_{\text{rel}} = 4300 \text{ m s}^{-1}$$

We can now solve the rocket equation to get the velocity of the rocket. Gravity will oppose the thrust, and we can add this in as a constant vector $\vec{g}$. In reality $\vec{g}$ will vary with height, but for simplicity we'll assume it is constant. We're also neglecting air resistance. Using vectors, we have the following

$$m\frac{d\vec{v}}{dt} = \vec{v}_{\text{rel}}\frac{dm}{dt} + m\vec{g}$$

We can integrate this to get the velocity of the rocket as a function of time

$$\int_{\vec{v}_0}^{\vec{v}} d\vec{v}' = \vec{v}_{\text{rel}} \int_{m_0}^{m} \frac{dm'}{m'} + \vec{g} \int_0^t dt'$$

$$\vec{v} - \vec{v}_0 = \vec{v}_{\text{rel}} \left[\log_e m'\right]_{m_0}^{m} + \vec{g}t$$

Rocket velocity

$$\vec{v} = \vec{v}_0 + \vec{v}_{\text{rel}} \log_e\left(\frac{m}{m_0}\right) + \vec{g}t$$

Example

Let's revisit the rocket example above. If the rocket started off with a mass $m_0 = 5.5 \times 10^5$ kg, what is its velocity after 200 seconds when the first stage burns out?

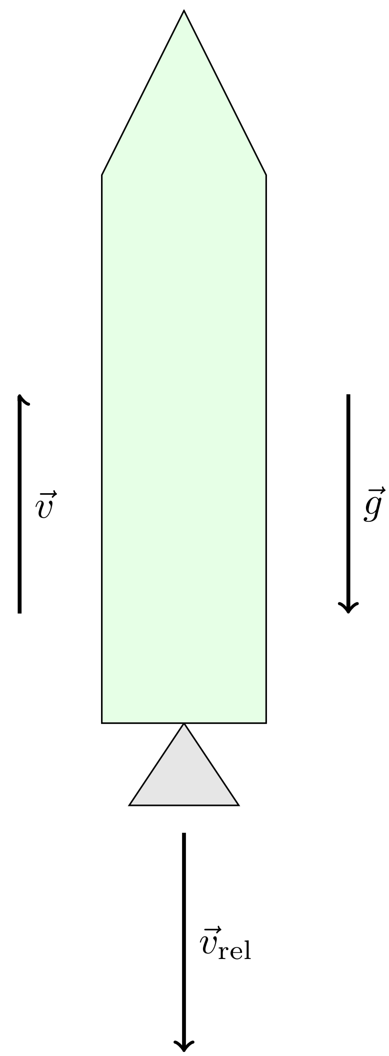

Figure 5.4: Rocket heading upwards showing the relative directions of the rocket velocity, exhaust velocity, and gravity.

Answer

- The initial velocity is zero, $\vec{v}_0 = \vec{0}$.
- $\vec{v}_{\text{rel}}$ and $\vec{g}$ are in the negative direction.
- $\vec{v}$ is in the positive direction.

This is illustrated in Figure 5.4.
The initial mass is $m_0 = 5.5 \times 10^5$ kg but 400,000 kg of fuel have been burned. This

means that the mass

$$m = 5.5 \times 10^5 - 4.0 \times 10^5 = 1.5 \times 10^5 \text{ kg}.$$

We calculated above that $|\vec{v}_{\text{rel}} = 4300$ m s^{-1}. Using the rocket velocity equation and these values gives

$$v = 0 - 4300 \log_e \left(\frac{1.5}{5.5}\right) - 9.81 \times 200$$

$$v = 3600 \text{ m s}^{-1}$$

5.4 Scalar product

I think this is a good time to have a mathematical interlude to consider the *scalar product* of two vectors. This is also known as the *dot product*. This takes two vectors and produces a scalar. It is a very useful operation in mechanics because it allows us to project vectors, e.g., forces, onto *any* direction specified by *any* unit vector.

So far, when resolving vectors (e.g., forces) we are used to projecting on the x and y axes (see Figure 5.5, in 2D). We can project these forces in the x and y directions as follows.

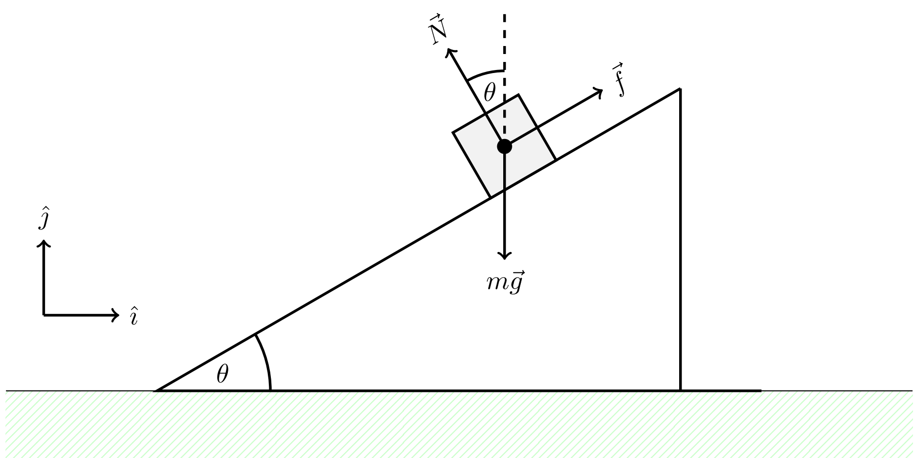

Figure 5.5: Projecting forces.

The components of $\vec{f}$ in the x and y directions can be seen in vector form,

$$\vec{f} = f \cos\theta \hat{\imath} + f \sin\theta \hat{\jmath}$$

The components of $m\vec{g}$ are

$$0\hat{\imath} - m|\vec{g}|\hat{\jmath} = -mg\hat{\jmath}$$

The components of $\vec{B}$ are

$$-N \sin\theta \hat{\imath} + N \cos\theta \hat{\jmath}$$

However, we can project forces in *any* direction specified by *any* unit vector.

For example, in 2D, we can write the components of $\vec{F}$ and unit vector $\hat{u}$ relative to an arbitrary set of axes. The component of $\vec{F}$ in the direction of $\hat{u}$ is $|\vec{F}| \cos\theta$ where θ is the angle between $\vec{F}$ and $\hat{u}$, as shown in Figure 5.6.

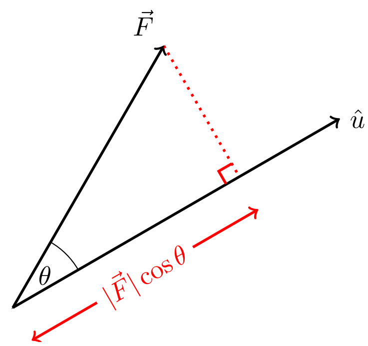

Figure 5.6: Projection of a vector $\vec{F}$ in the direction of unit vector $\hat{u}$.

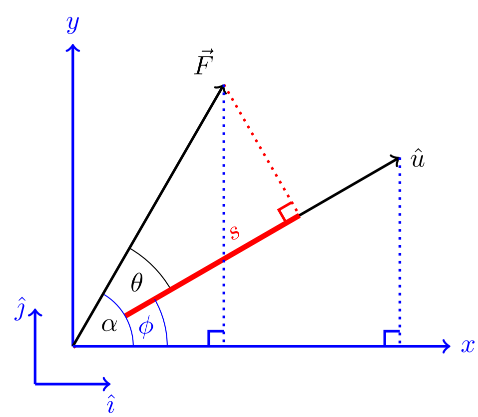

Figure 5.7: We can then find an expression for this projection using the components of the vectors in a particular coordinate system.

Using these axes, we have

$$\hat{u} = \cos\phi \hat{\imath} + \sin\phi \hat{\jmath} = \bar{u}_x \hat{\imath} + \bar{u}_y \hat{\jmath}$$

$$\vec{F} = F \cos\alpha \hat{\imath} + F \sin\alpha \hat{\jmath} = F_x \hat{\imath} + F_y \hat{\jmath}$$

Projecting $\vec{F}$ onto $\hat{u}$ gives a scalar, s,

$$s = |\vec{F}| \cos\theta$$

From Figure 5.7 we can see that $\theta = \alpha - \phi$ so

$$s = |\vec{F}|\cos(\alpha - \phi)$$

which we can expand using a trigonometric identity to get

$$\begin{aligned} s &= \left|\vec{F}\right|(\cos\alpha\cos\phi + \sin\alpha\sin\phi) \\ &= F_x\bar{u}_x + F_y\bar{u}_y \end{aligned}$$

We say that s is the *scalar product* (or dot product) of $\vec{F}$ with unit vector $\hat{u}$

$$s = \vec{F}\cdot\hat{u} = |\vec{F}|\cos\theta = F_x\bar{u}_x + F_y\bar{u}_y$$

Geometrically, this is the *projection* of $\vec{F}$ in the direction $\hat{u}$.

More generally

$$\vec{F}\cdot\hat{u} = \frac{\vec{F}\cdot\vec{u}}{|\vec{u}|}$$

from which we get

$$\vec{F}\cdot\vec{u} = |\vec{F}||\vec{u}|\cos\theta$$

Scalar product

The *scalar product* of *any* two vectors $\vec{F}$ and $\vec{u}$ in three dimensions is

$$\vec{F}\cdot\vec{u} = |\vec{F}||\vec{u}|\cos\theta = F_x u_x + F_y u_y + F_z u_z$$

where θ is the angle between $\vec{F}$ and $\vec{u}$.

Some special cases:

- The dot product of a vector with itself gives its magnitude squared $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2$.
- The dot product of two perpendicular vectors is zero. If $\vec{a}\perp\vec{b}$ then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 90 = 0$.

5.5 Forces

In discussing golf balls and rockets we have already been considering contact forces and force fields. Now we're going to look at these more carefully, and in more detail.

5.5.1 Contact forces

Contact forces act at a point on a body, generally due to contact with another part of the system. Typical examples of contact forces include the following.

- Friction acting between a sliding block and table.
- Tension in a rope.
- Push from a bat hitting a ball.

5.5.2 Force fields

Force fields generate forces that are a function of position (or velocity) only. Examples of force fields include the following.

- Force on a charge due to an electric field
- Gravitational force

The force on a charge q in an *electric field* $\vec{E}(\vec{r})$ is given by

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$$

where $\vec{r}$ is the position vector

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

.

The force of one mass on another due to their *gravitational field* is given by

$$F_{12} = \frac{Gm_1m_2}{r_{12}}$$

where r_{12} is the distance between the two masses. This is illustrated in Figure 5.8 which shows the force on m_1 due to m_2. The force on m_2 due to m_1 is equal and opposite. This is Newton's law of gravity.

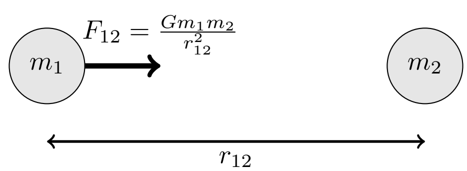

Figure 5.8: Force due to gravitational field.

5.6 Statics

We'll start by looking at some classic examples of *statics* problems where all of the forces are balanced so there is no net acceleration. This is also known as *equilibrium*. If we're in an inertial frame in which the object is at rest, it will remain at rest. We have

$$\sum_i \vec{F}_i = \vec{0}$$

where $\vec{F}_i$ are the individual forces acting on the body.

5.6.1 Example 1: a block on a table

A stationary block on a table is shown in Figure 5.9. The forces acting on the block are the weight $m\vec{g}$, the normal reaction force $\vec{N}$, which must balance if there is no acceleration.

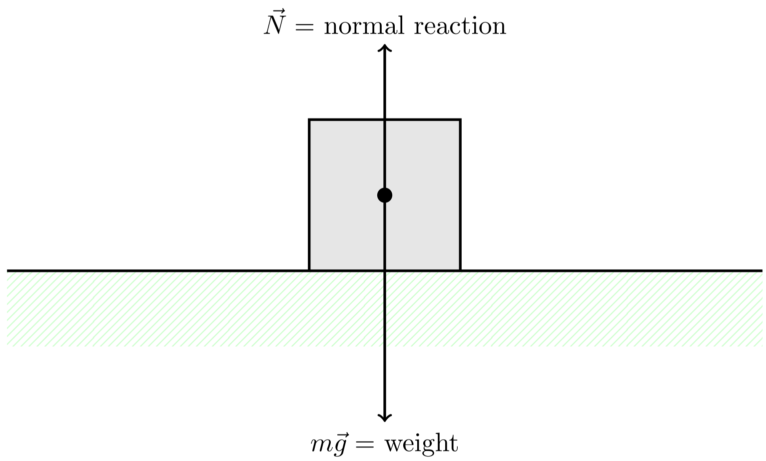

Figure 5.9: A static block at rest on a table.

$$\sum \vec{\text{forces}} = \vec{0}$$
$$m\vec{g} + \vec{N} = \vec{0}$$
$$m\vec{g} = -\vec{N}$$

so the vectors $\vec{g}$ and $\vec{N}$ are antiparallel, with $|\vec{N}| = m|\vec{g}|$.

5.6.2 Example 2: hanging a picture

A picture is hanging from two wires, as shown in Figure 5.10. Consider the forces acting through the centre of mass to avoid rotations (we'll discuss rotations later). We will consider the downwards direction to be "positive".

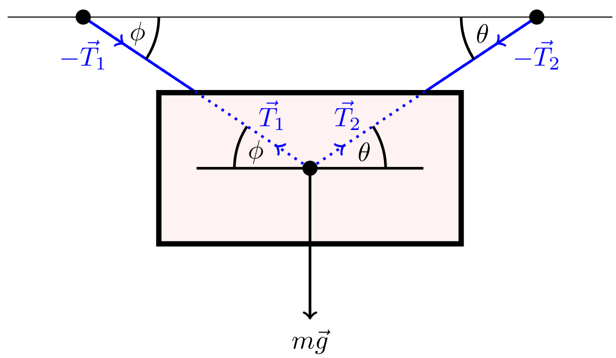

Figure 5.10: A static picture hanging form two wires.

The sum of the forces acting on the frame must be zero.

$$m\vec{g} + \vec{T}_1 + \vec{T}_2 = \vec{0}$$

where $\vec{T}_1$ and $\vec{T}_2$ are the tensions in the wires.

Consider the vertical component (obtained by taking the scalar product with $\hat{\jmath}$)

$$mg - T_1 \sin\phi - T_2 \sin\theta = 0 \tag{5.2}$$

The horizontal component (obtained by taking the scalar product with $\hat{\imath}$) is

$$-T_1 \cos\phi + T_2 \cos\theta = 0 \tag{5.3}$$

We can eliminate T_2 by taking (Equation 5.2) $\times \cos\theta +$ (Equation 5.3) $\times \sin\theta$ to get

$$mg\cos\theta - T_1 \sin\phi\cos\theta - T_1\cos\phi\sin\theta = 0$$

$$T_1 = \frac{mg\cos\theta}{\sin\phi\cos\theta + \cos\phi\sin\theta} = \frac{mg\cos\theta}{\sin(\theta+\phi)}$$

Similarly,

$$T_2 = \frac{mg\cos\phi}{\sin(\theta+\phi)}$$

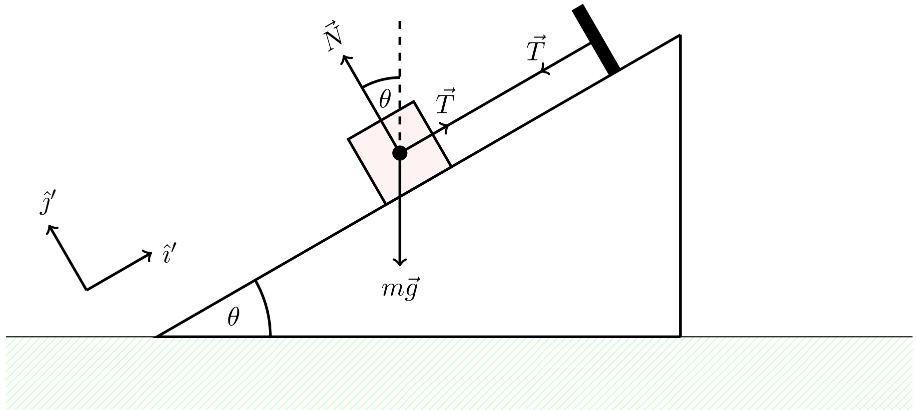

Figure 5.11: Tethered block

5.6.3 Example 3: tethered block on smooth plane

A block of mass m is tethered to a point on a smooth (frictionless) plane, as shown in Figure 5.11. The tension in the rope is $\vec{T}$ and the normal reaction force is $\vec{N}$.

The block is static so is not accelerating, i.e,. $\vec{a} = \vec{0}$. The sum of the forces acting on the block must therefore be zero,

$$m\vec{g} + \vec{N} + \vec{T} = \vec{0} \tag{5.4}$$

Note that we can choose coordinate axes parallel and perpendicular to the plane to simplify the maths. These coordinate unit vectors are shown in Figure 5.11 as $\hat{\imath}'$ and $\hat{\jmath}'$. In this coordinate system the individual force vectors are as follows.

$$m\vec{g} = -mg(\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}')$$
$$\vec{N} = N\hat{\jmath}'$$
$$\vec{T} = T\hat{\imath}'$$

Taking the scalar product of Equation 5.4 with $\hat{\imath}'$ and $\hat{\jmath}'$ gives the following scalar equations.

Parallel to the plane

$$-mg\sin\theta + T = 0$$
$$\therefore T = mg\sin\theta$$

Perpendicular to the plane

$$-mg\cos\theta + N = 0$$
$$\therefore N = mg\cos\theta$$

This analysis is simpler than if we have used the coordinate axes $\hat{\imath}$ and $\hat{\jmath}$. These still work, which you can try to confirm.

5.7 Dynamics

When the net forces are not balanced we have a non-zero acceleration.

$$\sum \overrightarrow{\text{forces}} = m\vec{a}$$

where $\vec{a}$ is the resultant acceleration of the body.

5.7.1 Example 1: block on an inclined smooth plane

Consider a block on a smooth, inclined plane as shown in Figure 5.12. The block has mass m and the angle of the plane is θ. The block is accelerating down the plane with acceleration $\vec{a}$.

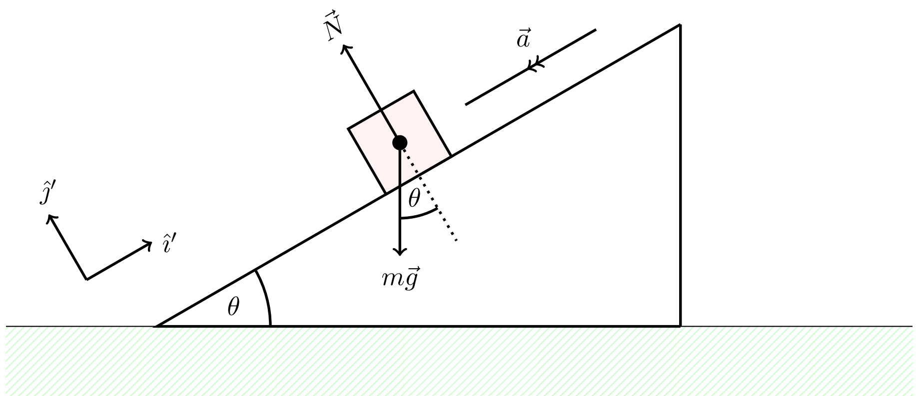

Figure 5.12: A block on an inclined smooth plane. The forces are not balanced so the block has an acceleration $\vec{a}$.

$$\sum F_i = m\vec{g} + \vec{N} = m\vec{a}$$
$$-mg(\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}') + N\hat{\jmath}' = ma\hat{\imath}'$$

The component parallel to the plan can be found by taking the scalar product with $\hat{\imath}'$,

$$-mg\sin\theta = ma$$

$$\therefore a = -g\sin\theta$$

The component perpendicular to the plane can be found by taking the scalar product with $\hat{\jmath}'$,

$$-mg\cos\theta + N = 0$$
$$N = mg\cos\theta$$

The acceleration vector is

$$\vec{a} = -g\sin\theta\hat{\imath}'$$

5.7.2 Example 2: block and spring

Consider a block on a horizontal, smooth table attached to a spring, as shown in Figure 5.13.

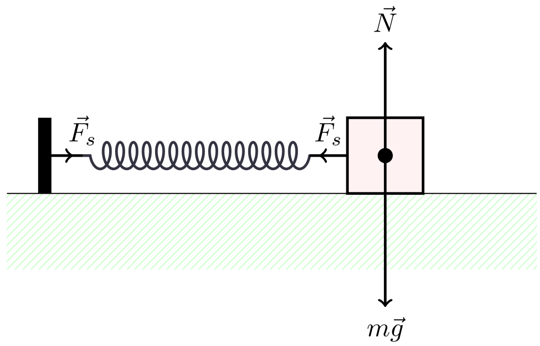

Figure 5.13: A block on a horizontal, smooth table attached to a spring.

As ususal,

$$\vec{F}_s + \vec{N} + m\vec{g} = m\vec{a}$$

The vertically the forces balance so there is no acceleration in the vertical direction, i.e., $\vec{N} = -m\vec{g}$.

Horizontally,

$$F_s = ma$$

The force provided by the spring is given by *Hooke's law*,

$$F_s = -kx$$

where k is the stiffness of the spring, and x is the extension of the spring. This gives us an equation of motion

$$m\ddot{x} + kx = 0 \tag{5.5}$$

where $\ddot{x}$ is the second derivative of x with respect to time.

This is the equation of simple harmonic motion (more about that in the third part of this unit on Oscillations and Waves). Some of you will have seen this before, but don't worry if you haven't. You can verify by substitution that the following is a solution to this equation

$$x = x_0 \cos(\omega t)$$

where x_0 s the initial extension of the spring, and

$$\omega = \sqrt{\frac{k}{m}}$$

is the *angular frequency*.

You should substitute this solution into Equation 5.5 to confirm.

5.8 Friction

Newton's first law states that bodies only stop *if a force acts*. However, we experience things stopping all the time. We now know that this stopping is cause by *friction*.

A classic example of friction is a block on a table. This is illustrated in Figure 5.14.

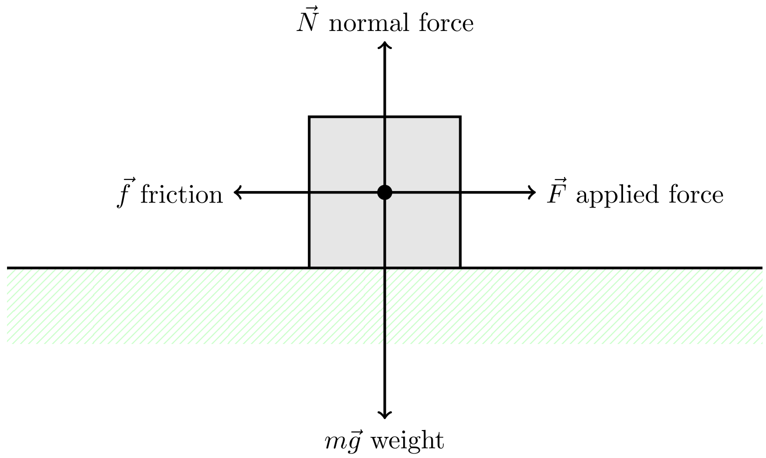

Figure 5.14: A block on a table subject to an applied force and friction opposing its motion.

We can use Newton's second law to write

$$\text{sum of forces acting} = \text{mass} \times \text{resultant acceleration of block}$$

Using vector notation this can be written

$$\vec{f} + \vec{F} + m\vec{g} + \vec{N} = m\vec{a}$$

We can look at the horizontal and vertical components of the motion separately.

The horizontal, scalar components are

$$F - f = ma$$

and the vertical components are balanced (no vertical acceleration)

$$N - mg = 0$$

$$\therefore N = mg$$

The horizontal acceleration of the block will depend on the applied force and the friction force.

5.8.1 Coefficient of friction

The friction force is difficult to model in detail because it depends on the surface roughness and composition. However, in experiments we empirically observe that the friction force is proportional to the normal force, i.e. the force perpendicular to the surface.

Coefficient of friction

$$|\vec{f}| \propto |\vec{N}|$$

In scalar notation

$$f = \mu N$$

in scalars where μ = *coefficient of friction*.

The friction force, $\vec{f}$, acts to oppose the direction of motion that would occur in the absence of friction.

5.8.2 Kinetic friction

If the object is moving, the friction force is called *kinetic friction*. The coefficient of kinetic friction is denoted μ_k. This is also known as sliding friction. Consider a block sliding along a rough table as shown in Figure 5.15.

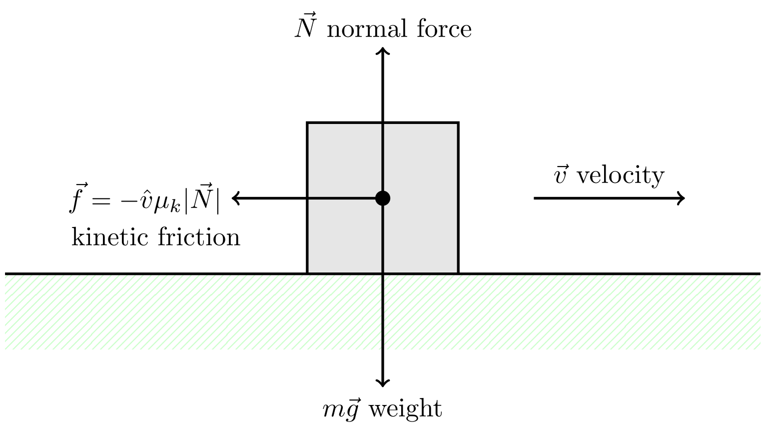

Figure 5.15: Kinetic friction for a block sliding along a rough table.

The friction force $\vec{f}$ is given by

$$\vec{f} = -\hat{v}\mu_k|\vec{N}|$$

where $\hat{v}$ is the unit vector in the direction of motion, and $|\vec{N}|$ is the magnitude of the normal reaction force. Notice that the friction force is in the opposite direction to the motion.

5.8.3 Static friction

If there is no net motion, the friction force is called *static friction*. The coefficient of static friction is denoted μ_s. This is also known as sticking friction. Consider a block on a rough table that is being pushed, but is not moving, as shown in Figure 5.16.

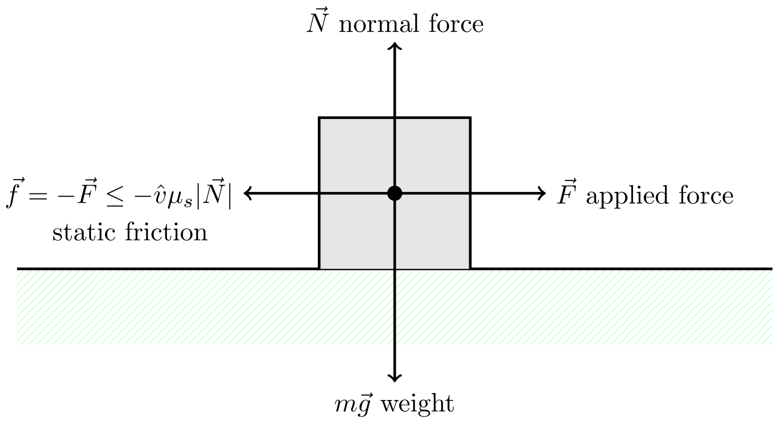

Figure 5.16: Static friction for a block on a rough table that is being pushed, but is not moving.

The friction force is given by

$$f \leq \mu_s|\vec{N}|$$

As the applied force $\vec{F}$ increases, the static friction force will also increase to balance it, i.e., $\vec{f} = -\vec{F}$. The friction force can increase to the limit $f = \mu_s|\vec{N}|$, beyond which the block will start to move, where the friction force is then equal to the (lower) kinetic friction force $f = \mu_k|\vec{N}|$.

In practice, $0 < \mu_k < \mu_s < 1$.

Typical values of μ_s and μ_k are given in Table 5.1.

Table 5.1: Coefficients of friction

Surface	μ_s	μ_k
Rubber on concrete	1.0	0.8
Steel on steel	0.7	0.6
Waxed ski	0.1	0.05

5.8.4 Drag

Friction due to motion through a fluid, gas or liquid is called *drag*. This is a major topic in fluid mechanics. The drag direction opposes the motion $\hat{f} = -\hat{v}$, as shown in Figure 5.17.

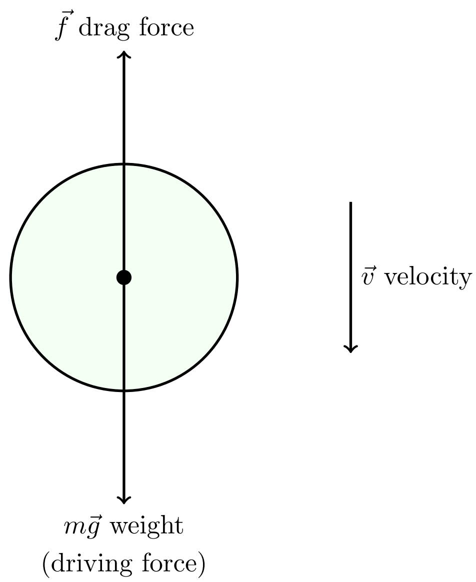

Figure 5.17: Drag force $\vec{f}$ acting on a sphere falling through a fluid due to gravitational force $m\vec{g}$. The velocity of the ball is $\vec{v}$.

Generally

$$|\vec{f}| \propto |\vec{v}|^n$$

- $n = 1$ for microscopic particles in water.
- $n = 1$ for human swimming in treacle.
- $n = 2$ for free-falling through the atmosphere.

6 Work and Energy

Work, Energy, and Momentum

When a force acts on a particle, such that its point of action moves through a distance, d, we say that work is done. How much work is done, though? Let's look at progressively more general examples.

6.1 Constant force parallel to displacement

This is a one dimensional problem. The *work done*, W, is given by

$$W = Fd$$

which is the force, F, multiplied by the displacement, d.

Power is the rate of doing work, which we can get by differentiating the work done with respect to time.

$$P = \frac{dW}{dt} = F\frac{d(d)}{dt} = Fv$$

where v is the velocity of the particle. (The force, F is constant so it can be taken outside the derivative).

Note that these expressions also make sense dimensionally.

6.2 Varying force parallel to displacement

If the force varies with position, i.e., force is a function of position $F(x)$, then we can still calculate the work done by dividing the path into elements of length δx_i over which the force is (almost) constant, $F(x_i)$, where x_i is the position of the i^{th} element. The work done by the force moving along each element is then

$$W_i = F(x_i)\delta x_i$$

The total work done is then the sum of all the elements

$$W = \sum_i W_i = \sum_i F(x_i)\delta x_i$$

If we take the limit as $\delta x_i \to 0$, then we get an integral

$$W = \int_{x=0}^{x=d} F(x)dx$$

where $x = 0$ and $x = d$ are the initial and final positions of the particle.

If we know how $F(x)$ varies we can calculate the work done.

6.3 Force not parallel to displacement

If the force is not parallel to the displacement, we can still calculate the work done by considering the component of the force parallel to the displacement. If the force makes an angle θ with the displacement, then the component of the force parallel to the displacement is $F\cos\theta$. The work done moving a distance d is then

$$W = Fd\cos\theta$$

where d is the displacement, as shown in Figure 6.1. But this is the definition of the scalar product of $\vec{F}$ and $\vec{d}$.

Work done

Work done, W, by a force $\vec{F}$ at an angle θ to the displacement $\vec{d}$ is given by the scalar product of $\vec{F}$ and $\vec{d}$,

$$W = \vec{F} \cdot \vec{d}$$

6.4 Example 1: block sliding down a frictionless plane

Consider a block, initially at rest, sliding down a frictionless plane. This is illustrated in Figure 6.2.

The component of the gravitational force parallel to the plane is

$$mg\sin\theta$$

This means that the work done is this component of the force times the distance moved, d, given by

$$W = mgd\sin\theta$$

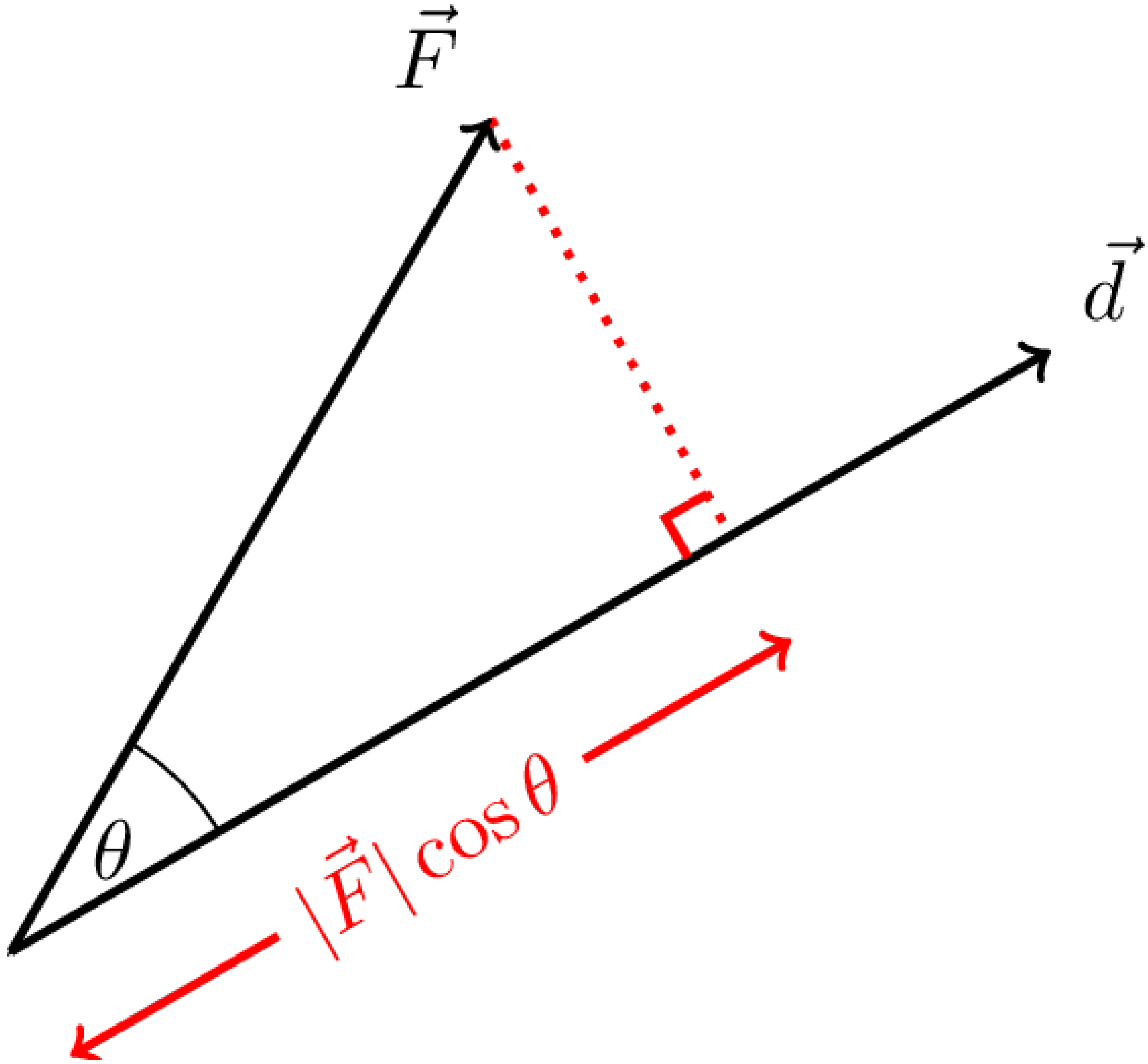

Figure 6.1: Work done, W, by a force $\vec{F}$ at an angle θ to the displacement $\vec{d}$ is given by $W = |\vec{F}||\vec{d}|\cos\theta = \vec{F}\cdot\vec{d}$.

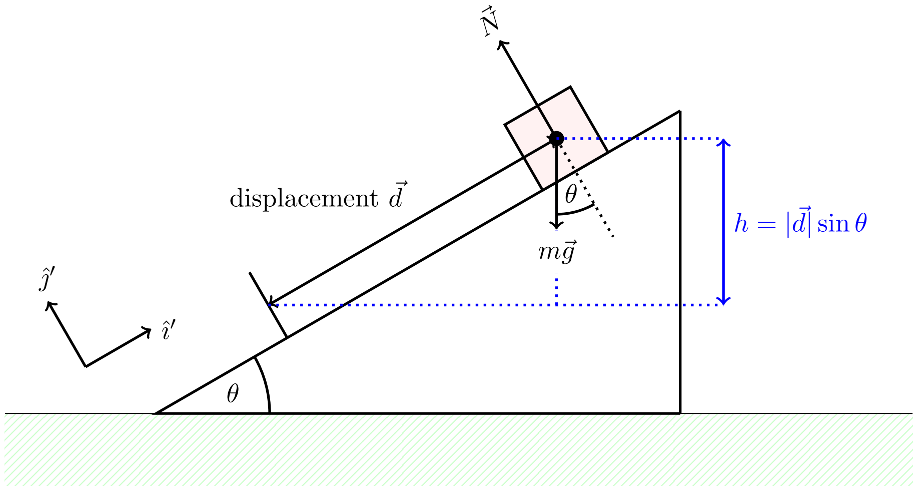

Figure 6.2: A block of mass m sliding down a frictionless plane.

Interestingly, the work done exactly matches the loss of gravitational potential energy, mgh, where $h = d \sin\theta$ is the vertical distance the block falls through (see Figure 6.2). This is not a coincidence, as we will see later.

6.5 Example 2: circular motion

Consider a particle moving in a circle orbit, as shown in Figure 6.3. The centripetal force is always perpendicular to the motion. The instantaneous displacement, $d\vec{s}$, is parallel to $\vec{v}$ because $\vec{v} = d\vec{s}/dt$.

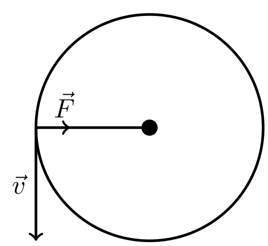

Figure 6.3: A particle moving in a circular orbit.

This means that $\vec{F} \cdot \vec{ds} = 0$ so *no work* is done.

6.6 Kinetic energy

Consider a force moving through a displacement d, in one dimension. The work done, W is given by

$$\Delta W = \int_0^d F(x)dx$$

The force gives rise to an acceleration, which results in a change of velocity (or speed in one dimension).

If we assume the initial velocity $= 0$, and the final velocity $= v_f$, we can compare this with the work done as follows.

$$\Delta W = \int_0^d ma(x)dx$$

$$\Delta W = \int_0^d mv\frac{dv}{dx}dx = \int_0^{v_f} mvdv$$

where we have used the cain rule to write $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$.

$$\Delta W = \left[\frac{mv^2}{2}\right]_0^{v_f} = \frac{1}{2}mv_f^2$$

which is the *kinetic energy* associated with the motion of the particle.

Therefore, *work done* produced *kinetic energy*.

If the initial velocity $= v_0$, we would obtain

Kinetic energy

Work done $\Delta W =$ *change* in kinetic energy

$$\Delta W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

where v_i is the initial velocity, and v_f is the final velocity.

6.6.1 Example: kinetic friction

Consider a block sliding along a rough surface that is brought to rest from a speed v_i by a kinetic friction force of magnitude $\mu_k N$, where N is the magnitude of the normal reaction force. This is shown in Figure 6.4.

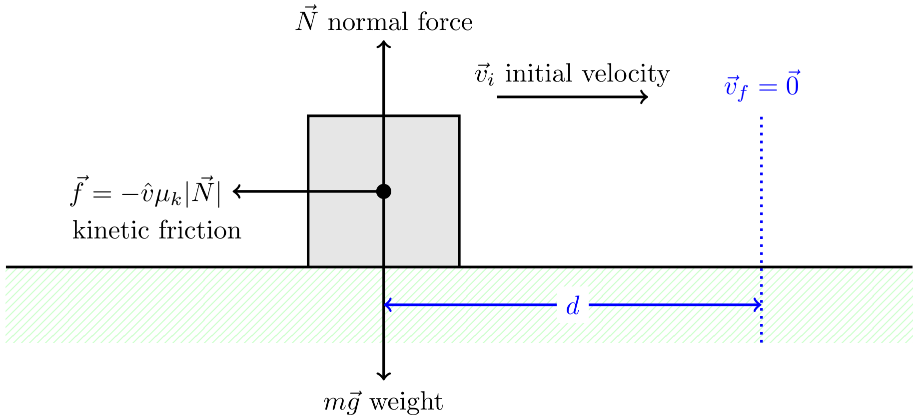

Figure 6.4: A block of mass m sliding along a rough surface being brought to rest by kinetic friction.

$$\Delta W = 0 - \frac{1}{2}mv_i^2$$
$$F = \mu_k N = \mu_k mg$$
$$\Delta W = \vec{F} \cdot \vec{d} = -Fd = -\mu_k mgd$$

$\Delta W < 0$ as particle *loses* kinetic energy. We can equate ΔW with the kinetic energy to get

$$\frac{1}{2}mv_i^2 = \mu_k mgd$$

$$\therefore v_i^2 = 2\mu_k gd$$

6.6.2 KE in different reference frames

Note that the kinetic energy is not the same in different reference frames, just as the velocity is not the same in different reference frames. This means that statements regarding conservation of energy *must* be restricted to a single reference frame.

6.7 Potential energy

If a force is a function of position (or a constant) we can define a *potential difference*, ΔV, of a particle, as the work done *against* the force in moving from one position, A, to another position, B. This is given, in one dimension, by

$$\Delta V = V(B) - V(A)$$

where

$$V(A) = -\int_0^A F(x)dx \tag{6.1}$$

where the negative sign is because the force is acting *against* the force. The integration is from zero here, but it could be some other reference point, e.g., $-\infty$, where the potential V is *defined* to be zero.

$$V(B) = -\int_0^B F(x)dx$$

Examples of forces that depend only on a particle's position include gravity and the electrostatic force.

$V(A)$ is a *scalar* function of position, known as the *potential*.

From Equation 6.1, we can see that a differential from will also hold,

$$F(x) = -\frac{dV(x)}{dx}$$

which is a direct consequence of the definition of the potential.

> **Potential**
>
> The force $F(x)$ is minus the gradient of the potential $V(x)$,
>
> $$F(x) = -\frac{dV(x)}{dx}$$

The zero of the potential is *not* fixed and depends on the (sometimes arbitrary) choice of reference point. Adding a constant to the potential does *not* change the gradient or, therefore, the force. The derivative of a constant is zero.

We can therefore choose a convenient zero point for the potential, appropriate to the problem being studies.

6.7.1 Example 1: lifting a particle against gravity

Consider lifting a particle through a height h, doing work against gravity, as shown in Figure 6.5.

The potential difference is given by

$$\Delta V = V(h) - V(0)$$

$$\Delta V = -\int_0^h (-mg)dz = mgh$$

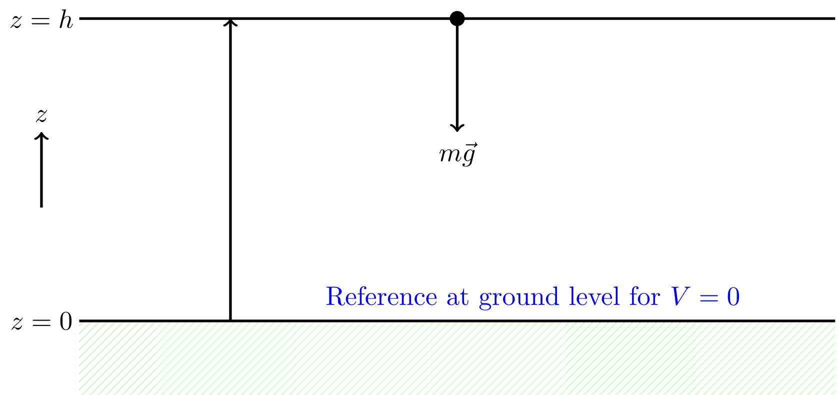

Figure 6.5: A particle of mass m being lifted through a height h against gravity. We are free to set the zero point for the potential at ground level for convenience.

Note the minus sign because we've defined positive z to be up.

$$\therefore V(h) = mgh + V(0)$$

where $V(0)$ is the potential at $z = 0$ which we can *choose* to be our reference level with a potential of zero.

6.7.2 Example 2: particle escaping gravity

Consider a particle escaping form the surface of a planet, as illustrated by Figure 6.6.

Newton's law of gravity states that the force between the particle of mass m and the planet of mass M is given by

$$F = -\frac{GMm}{r^2}$$

where r is the distance between the particle and the centre of the planet, and G is the gravitational constant. To escape gravity we need to take the particle from the surface at radius R to ∞. The work done against the force of gravity will be

$$\Delta V = V(\infty) - V(R)$$

$$\Delta V = \int_R^\infty \left(-\frac{GMm}{r^2}\right) dr = GMm \left[-\frac{1}{r}\right]_R^\infty$$

$$\therefore \Delta V = \frac{GMm}{R}$$

for escape to infinity, i.e., the particle needs to gain this energy.

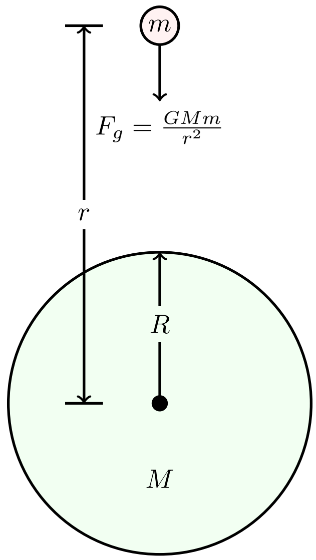

Figure 6.6: Consider a particle of mass m at a distance r from the centre of a planet of mass M and radius R that is escaping to infinity.

6.7.3 Example 3: horizontal stretched spring

We can caluclate the work done by moving the block and stretching the spring from 0 to x, as show in Figure 6.7.

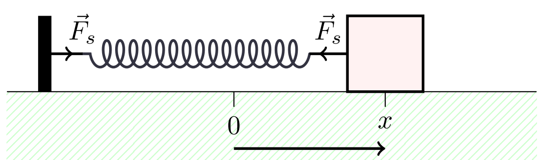

Figure 6.7: Work done stretching a spring from 0 to x. The force from the spring follows Hooke's law, so $\vec{F}_s = -k\vec{x}$.

$$\Delta V = V(x) - V(0)$$
$$\Delta V = -\int_0^x (-kx')dx' = \frac{1}{2}kx^2$$

where k is the spring constant.

$$\therefore V(x) = \frac{1}{2}kx^2 + V(0)$$

Normally choose $V(0) = 0$, i.e., origin at equilibrium position.

6.8 Conservation of energy

The mechanical energy we've looked at so far can be divided into two parts: *kinetic energy* due to the motion, and *potential energy* due to the position. The total mechanical energy is the sum of the kinetic and potential energies,

When work is done by, e.g., friction or drag, energy is lost mainly as heat, sound, etc.

> Conservation of energy
>
> The *total energy* of a *closed* system is constant in time. Energy can be converted from one form to another, but not created or destroyed.

By a *closed system* we mean one that is not acted on by external forces. This is a *very* important point. If external forces are acting on the system, then the total energy of the system is *not* conserved.

If there *is* an external force, the total force acting on the particle is written as

$$F_{\text{ext}} + F(x) = ma$$

Therefore the work done by the external force can be written as

$$\Delta W_{\text{ext}} = \int_{x_0}^{x_1} F_{\text{ext}} dx = \int_{x_0}^{x_1} ma dx - \int_{x_0}^{x_1} F(x) dx$$

From before we have that

$$\int_{x_0}^{x_1} ma dx = \int_{x_0}^{x_1} mv \frac{dv}{dx} dx = \frac{1}{2} m (v_1^2 - v_0^2) = \Delta K$$

and

$$-\int_{x_0}^{x_1} F(x) dx = V(x_1) - V(x_0) = \Delta V$$

so

$$\Delta W_{\text{ext}} = \Delta K + \Delta V$$

where ΔK is the change in kinetic energy, and ΔV is the change in potential energy. I.e., external force leads to changes in both KE and PE.

Equally, if external force is zero, total change in KE and PE is zero,

$$0 = \Delta K + \Delta V$$

This is *conservation of energy.*

6.8.1 Example: escape velocity

Consider a particle with *just* enough kinetic energy to escape to infinity from the surface of a planet (ignoring resistance from the atmosphere of the planet). The initial configuration is shown in Figure 6.8.

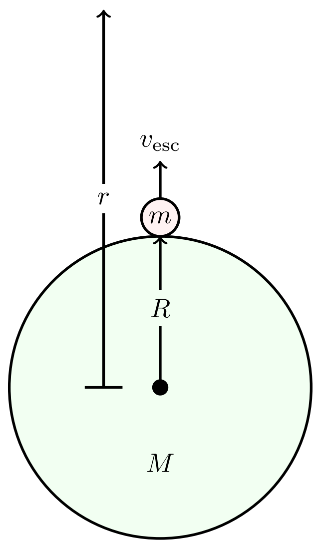

Figure 6.8: A particle of mass m with initial velocity v_{esc} at a distance R from the centre of a planet of mass M that has *just* enough kinetic energy to escape to infinity.

If the particle *just* escapes to infinity, this means that the velocity $v \to 0$ as $r \to \infty$. The final total energy is zero, if we define the zero point of the gravitational potential to be at infinity (see Figure 6.9). We can then use conservation of energy

$$\underbrace{0}_{\text{final energy}} = \underbrace{-\frac{GMm}{R}}_{\text{initial PE}} + \underbrace{\frac{1}{2}mv_{\text{esc}}^2}_{\text{initial KE}}$$

$$\therefore \quad v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = 11.2 \text{ km s}^{-1} \text{ for Earth.}$$

Thought experiment

What would happen if the planet was shrunk so that $R = GM/c^2$?

6.8.2 Proof of conservation of energy

This isn't examinable but is included for those interested in seeing the proof.

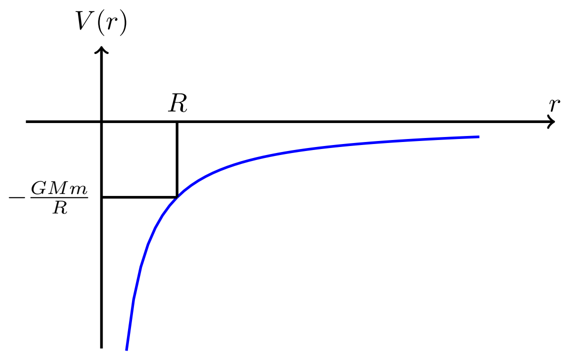

Figure 6.9: A plot of the gravitational potential $V(r)$ as a function of r.

Proof of conservation of energy

Let the total energy of a mechanical system be

$$E_{\text{tot}} = \frac{1}{2}mv^2 + V(x)$$

The total energy, E_{tot} depends on arbitrary choices of the inertial frame which affects KE and the choice of the zero point of the potential $V(0)$ which affects PE.
If E_{tot} is conserved, it means that it is not changing with time, i.e.,

$$\frac{dE_{\text{tot}}}{dt} = 0$$

This means

$$\frac{dE_{\text{tot}}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + V(x)\right) = 0$$

Applying the chain rule to both terms,

$$\frac{dE_{\text{tot}}}{dt} = \frac{1}{2}m2v\frac{dv}{dt} + \frac{dV(x)}{dx}\frac{dx}{dt}$$

$$= mva + v \times \underbrace{\frac{dV(x)}{dx}}_{\text{gradient of potential}=-F}$$

$$\therefore \frac{dE_{\text{tot}}}{dt} = v\,(ma - F) = 0$$

by Newton's second law.
So, in the absence of external forces, energy is conserved.
Notes

- Newton's first law: total energy depends on reference frame, but energy is conserved in all reference frames, if it is conserved in one.

- Newton's third law: must include both forces in an action-reaction pair, to avoid external forces
- In practice, for gravity problems including the Earth, the work done on the Earth is $\simeq 0$ because displacement is tiny
- Energy conservation often provides an alternative approach to Newton's laws for solving mechanics problems (and can often be easier!)

6.9 Conservative forces

Conservative forces can be written as minus the gradient of a potential. Consider the work done against a force $\vec{F}(\vec{r})$. We can only define a potential when

$$\underbrace{V(\vec{r}_1) - V(\vec{r}_0)}_{\text{depends only on end points}} = \underbrace{\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}(\vec{r}) \cdot d\vec{r}}_{\text{must be independent of path}}$$

Conservative forces

Forces $\vec{F}(\vec{r})$ for which a potential $V(\vec{r})$ can be defined are called *conservative forces*. The work done by a conservative force around a closed loop is zero.
In 1D

$$F = -\frac{dV}{dx}$$

In 3D

$$\vec{F}(\vec{r}) = -\text{gradient}(V(\vec{r})) = -\vec{\nabla}V(\vec{r})$$

where the gradient is defined in the following way

$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r}) = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k}$$

Gravity and electrostatic forces are examples of conservative forces.

Don't worry about the 3D maths and the path integral described above; that will be covered in maths units later. The important physics point is that for a *conservative force* the potential difference between two points A and B only depends on those points, it does not matter what path the particle took to get from A to B.

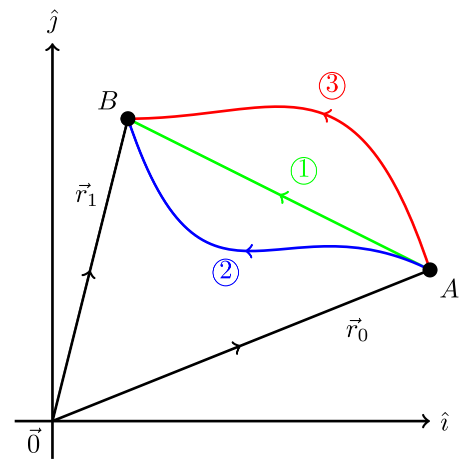

Figure 6.10: For a *conservative force* the potential difference $V(\vec{r}_1) - V(\vec{r}_0)$ for a particle moving from point A to point B depends only on the position of the end points. It does not matter what path the particle takes between these points (e.g., paths 1, 2, or 3 in the figure), $\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}(\vec{r}) \cdot d\vec{r}$ is the same. An example of a conservative force is gravity, in which case the potential difference between A and B would only depend on their height above ground, not the path taken between them.

6.10 Collisions with conservation of energy and momentum

We are going to consider collisions between particles in one frame of reference so we can consider energy to be conserved, as well as momentum, as we will be considering a closed system (i.e., no external forces). We'll start by considering a one-dimensional example.

Note that sometimes we can reduce a 2D problem to a 1D problem by working in the centre of mass frame, which can make things simpler. See, e.g., Figure 6.11. This works fine for point particles.

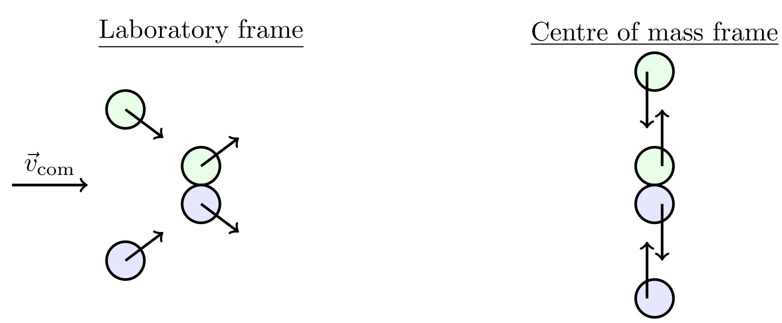

Figure 6.11: A 2D problem in the laboratory frame (left) can be transformed into a 1D problem in the centre of mass (com) frame (right). This is much simpler to solve.

6.10.1 Collision in 1D

Consider two particles colliding in 1D as shown in Figure 6.12.

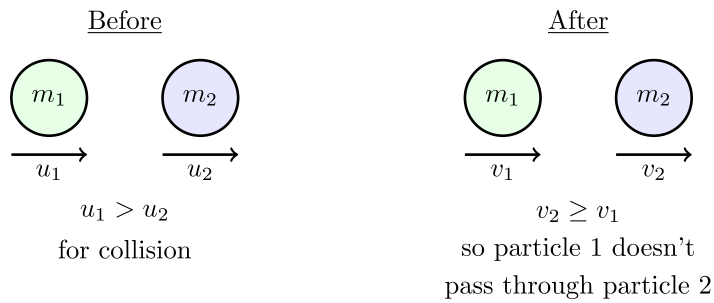

Figure 6.12: Collision between two particles in one-dimension.

Before the collision $u_1 > u_2$ otherwise the particles would not collide. After the collision $v_2 \geq v_1$ because the particles can't pass through each other (at least in *classical* mechanics).

Conservation of momentum gives us

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Note, however, that not all collisions conserve kinetic energy (in general some energy will be lost to heat, sound, etc.). We can quantify this by defining a *coefficient of restitution*.

Coefficient of restitution

The *coefficient of restitution*, e, is defined as the ratio of the relative velocity of separation to the relative velocity of approach,

$$e = \frac{\text{relative speed after collision}}{\text{relative speed before collision}}$$

$$e = \frac{|v_2 - v_1|}{|u_1 - u_2|}$$

$$0 \leq e \leq 1$$

These relative speeds are along the line of impact (best suited to 1D problems). There are two special cases

- $e = 1$ for a perfectly *elastic* collision
- $e = 0$ for a perfectly *inelastic* collision

Perfectly elastic collisions: If $e = 1$ we have perfectly *elastic* collisions, so $u_1 - u_2 = v_2 - v_1$, or $u_1 + v_1 = u_2 + v_2$. Conservation of momentum can be written at

$$m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

Multiplying these two expressions together (and then multiplying by 1/2) gives

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

i.e., *perfectly elastic* collisions result in *conservation of energy.*

Perfectly inelastic collisions: If $e = 0$ we have perfectly *inelastic* collisions, $v_2 = v_1 = v$, so the particles stick together. v will be equal to v_{com}, the velocity of the centre of mass.

If $v = 0$ the final KE = 0 which happens if $m_1u_1 = -m_2u_2$, i.e., if $v_{\text{com}} = 0$ before the collision because momentum is still conserved.

Centre of mass frame

If we are in the *centre of mass frame* then the total momentum is zero, both before and after the collision. Hence, in 1D (for simplicity) we have

$$m_1u_1 + m_2u_2 = 0$$

and

$$m_1v_1 + m_2v_2 = 0$$

The equation for the coefficient of restitution, also in 1D, is then

$$e = \frac{|v_2 - v_1|}{|u_2 - u_1|} = \frac{v_2 - v_1}{u_1 - u_2}$$

These can be combined, with some algebra, to give

$$\frac{\text{Total final KE}}{\text{Total initial KE}} = e^2$$

But this is *only* true in general in the *centre of mass frame.*

6.10.2 General momentum conservation

We can generalise our analysis beyond the one dimensional case, as shown in Figure 6.13. If we use vectors, each component of the momentum $\vec{p}$ is separately conserved.

We can write the conservation of momentum as

$$\vec{p}_{\text{before}} = \vec{p}_{\text{after}}$$

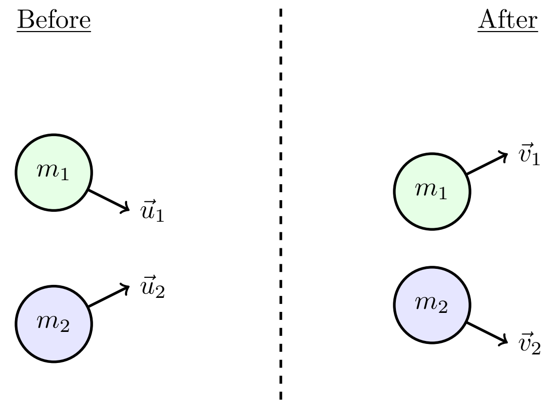

Figure 6.13: Collision between two particles illustrated in 2D.

This gives

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

which can be expanded to give three equations for conservation of energy in the x, y, and z directions. I.e.,

$$m_1u_{1x} + m_2u_{2x} = m_1v_{1x} + m_2v_{2x}$$

$$m_1u_{1y} + m_2u_{2y} = m_1v_{1y} + m_2v_{2y}$$

$$m_1u_{1z} + m_2u_{2z} = m_1v_{1z} + m_2v_{2z}$$

where u_{1x} is the x component of the velocity of particle 1 before the collision, etc. and momentum is separately conserved in each direction.

Also, if the collision is perfectly elastic, kinetic energy is conserved so

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

If you can reduce the problem to 1D, e.g., for two particles in the centre of mass frame, then the coefficient of restitution *can* be used.

7 Newton's third law

We're now going to introduce Newton's third law of motion.

> **Newton's third law**
>
> To any action there is an equal and opposite reaction.
> The actions of two bodies on each other are always equal and always opposite in direction.

7.1 Example: Normal reaction force

We need to be careful about identifying Newton's third law pairs. Consider a block resting on a table, as shown in Figure 7.1.

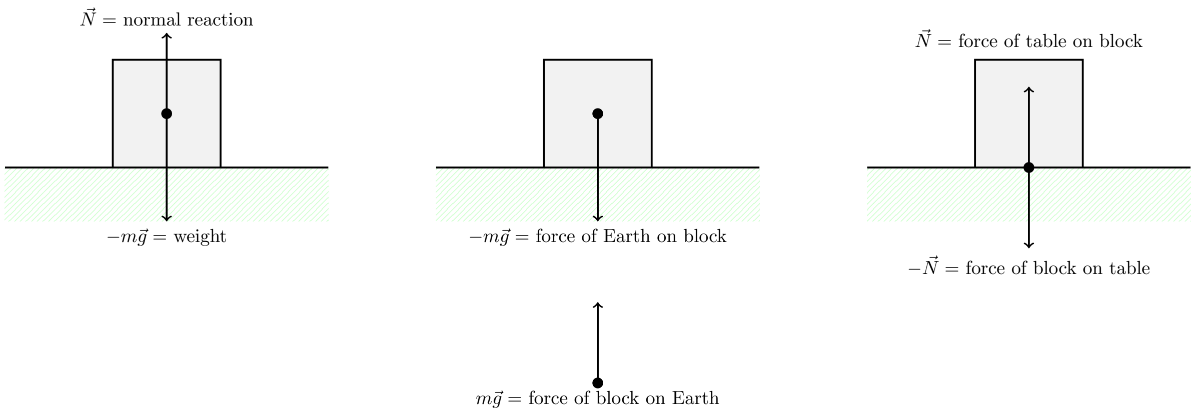

Figure 7.1: Normal reaction force between a block and table. The diagram shows the forces acting on the block and identifies the correct Newton's third law pairs.

The equal and opposite pairs of forces described by Newton's third law are as follows.

1. The gravitational interaction between the block and the Earth (*centre panel*)

- Earth pulls down on block with force $-m\vec{g}$
- Block pulls up on Earth with force $m\vec{g}$

2. The contact force at the interface between the block and the table (*right panel*)

- Table pushes up on block with force $\vec{N}$
- Block pushes down on table with force $-\vec{N}$

Consider the forces acting only on the block (*left panel*)

- The Earth exerts a gravitational force $-m\vec{g}$ downward on the block (weight)
- The table exerts a normal force $\vec{N}$ upward on the block

These two forces are equal in magnitude when the block is in equilibrium ($\sum \vec{F} = 0$), but they are *not* a Newton's third law pair because both forces act on the same object (the block) while Newton's third law pairs must act on different objects.

The weight and normal force happen to be equal because of Newton's second law (equilibrium condition), not the third law.

7.2 Example: Tension in rope

Consider a mass supported by a rope, as shown in Figure 7.2.

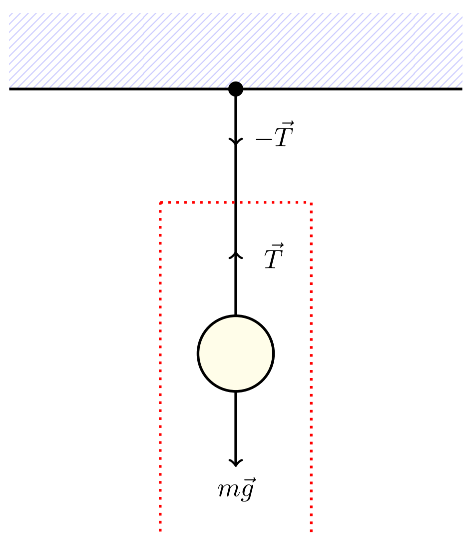

Figure 7.2: Tension in a rope. There are opposite and equal forces at either end of the rope.

- The weight induces tension in the rope which pulls on the support
- The support pulls back on the rope and weight
- These are balanced (i.e., equal and opposite) so there is no acceleration

Consider *only* the forces acting on the weight when analysing the weight's behaviour (outlined by the red box in Figure 7.2).

For the weight

$$m\vec{g} + \vec{T} = \vec{0}$$

7.2.1 Example: Bicycle wheel

Consider the bicycle wheel shown in Figure 7.3.

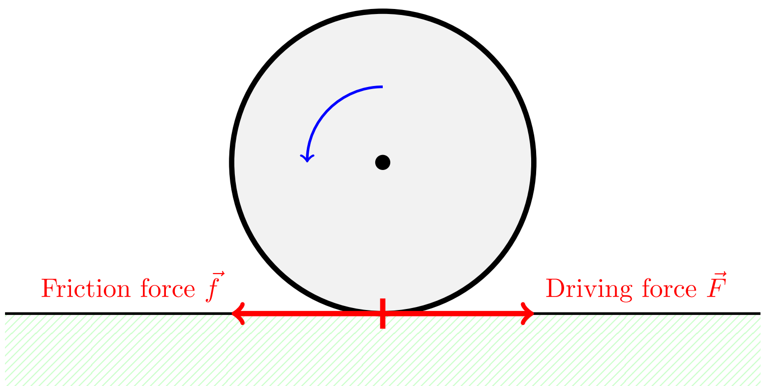

Figure 7.3: Bicycle wheel turning anti-clockwise. The wheel exerts a force $\vec{F}$ against the road, and the road exerts a friction force $\vec{F}$ against the wheel.

- The wheel exerts a force $\vec{F}$ against the road.
- The road exerts a friction force $\vec{f}$ against the wheel, and drives the bicycle forwards.
- $\vec{f} = -\vec{F}$

7.2.2 Example: Stepping off boat

Consider stepping off a boat onto the river bank, as shown in Figure 7.4.

The vertical forces balance, so let's look at the horizontal forces.

The foot pushes against the boat with force $\vec{F}$; the boat pushes back with friction force $-\vec{F}$, accelerating the person towards the bank, and the boat away.

The boat pushes against the water with force $\vec{f}$; the water pushes back against the boat with drag force $-\vec{f}$.

The forces acting on the boat are shown in the green box, namely $\vec{F}$ and $-\vec{f}$.

If there is insufficient drag, the boat can move greatly during the time taken to step off with the result being disaster!

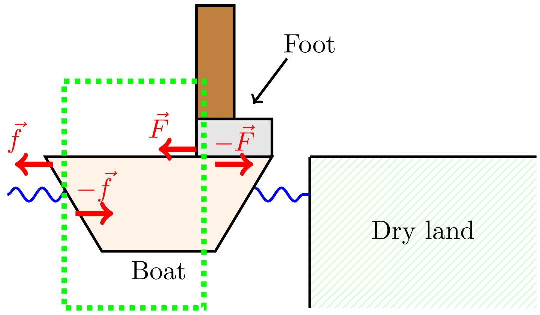

Figure 7.4: Stepping off a boat. Foot pushes against boat with force $\vec{F}$; boat pushes back with force $-\vec{F}$. Boat pushes against water with force $\vec{f}$ and water pushes back against boat with force $-\vec{f}$. The forces acting on the boat are shown in the green box.

Ultimately, the person stepping off the boat accelerates to the right, while the boat accelerates to the left. The respective accelerations will depend on their respective masses. This is an example of the *two-body problem*.

7.2.3 Two body problem and reduced mass

Consider a pair of particles interacting with each other, e.g., through gravity, as shown in Figure 7.5.

Figure 7.5: Two particles interacting with each other. The force on particle 1 from particle 2 is $\vec{F}_{12}$. The force on particle 2 due to particle 1 is $\vec{F}_{21}$.

The force on particle 1 due to particle 2 is $\vec{F}_{12}$. The force on particle 2 due to particle 1 is $\vec{F}_{21}$.

We can apply Newton's second law to both particles.

$$m_1\vec{a}_1 = \vec{F}_{12}$$

$$m_2\vec{a}_2 = \vec{F}_{21}$$

Then Newton's third law tells us that

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\therefore m_1\vec{a}_1 = -m_2\vec{a}_2$$

$$\vec{a}_2 = -\frac{m_1}{m_2}\vec{a}_1$$

From a frame in which particle 2 is at rest (*Note*: this is a non-inertial frame because the particle is accelerating), the net motion of m_1 is (see Figure 7.6)

$$\vec{a}_{\text{net}} = \vec{a}_1 - \vec{a}_2$$

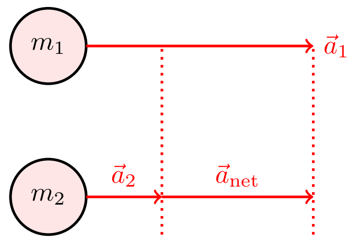

Figure 7.6: The net acceleration measured in the frame of particle 2 is $\vec{a}_{\text{net}} = \vec{a}_1 - \vec{a}_2$. *Note* that because particle 2 is accelerating this is *not* an inertial frame.

Note: previously we said that acceleration appears the same in *all* inertial frames. However, here we have *accelerating* frames, so this does not hold.

So, we have

$$\vec{a}_{\text{net}} = \vec{a}_1 + \frac{m_1}{m_2}\vec{a}_1$$

$$\vec{a}_{\text{net}} = \left(1 + \frac{m_1}{m_2}\right)\vec{a}_1 = \left(\frac{m_2 + m_1}{m_2}\right)\vec{a}_1$$

$$\vec{a}_{\text{net}} = \left(\frac{m_2 + m_1}{m_1 m_2}\right)\underbrace{m_1\vec{a}_1}_{\vec{F}_{12}}$$

In the frame of particle 2,

$$\vec{F}_{12} = \mu\vec{a}_{\text{net}}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass*.

Reduced mass

The equation of motion of particle 1 in the non-inertial frame of particle 2 is

$$\vec{F}_{12} = \mu\vec{a}_{\text{net}}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the *reduced mass*.

7.2.4 Example: two beads on a spring

Consider two beads on a spring. This could, for example, be a model for diatomic molecule. This is illustrated in Figure 7.7.

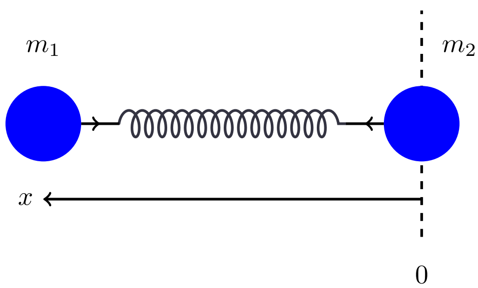

Figure 7.7: Two beads attached by a spring. We'll analyse this motion from a frame in which m_2 is at rest (this is *not* an inertial frame because it is accelerating).

Let's look at this motion in the frame of particle 2. In 2's frame, 1 is moving. The acceleration is given by

$$\ddot{\vec{x}} = \vec{a}_{\text{net}}$$

As above,

$$\vec{F}_{12} = \mu\vec{a}_{\text{net}} = -k\vec{x}$$

where k is the stiffness of the spring, and $\vec{x}$ is the extension.

In 1D,

$$a_{\text{net}} = \ddot{x} = -\frac{k}{\mu}x$$

which is the equation of motion of a simple harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{k}{\mu}}$$

This method effectively reduces the 2-body problem to a 1-body problem, in terms of relative separation, relative acceleration, and reduced mass.

Oxygen molecule bond stiffness

Continuing the diatomic molecule analogy, let's consider an Oxygen (O_2) molecule. Oxygen is observed to absorb infrared photons which can excite this harmonic oscillation with an angular frequency of

$$\omega = 2.94 \times 10^{14} \text{ rad s}^{-1}.$$

The mass of an Oxygen atom is $m = 2.66 \times 10^{-26}$ kg, so the reduced mass of the system is

$$\mu = 1.33 \times 10^{-26} \text{ kg}.$$

The bond stiffness can then be calculated using

$$\omega = \sqrt{\frac{k}{\mu}}$$

$$k = \mu\omega^2 = 1146 \text{ N m}^{-1}$$

7.3 Composite systems

These are systems with different connected components, e.g., blocks stacked or connected to one another, or components connected by ropes.

The following examples illustrate the pairs of opposite and equal forces that arrise in composite systems in accordance with Newton's third law.

7.3.1 Example: row of blocks

Consider a row of three blocks, each of mass m, as shown in Figure 7.8.

The force at the left, $3F$, is pushing three blocks that are in contact with one another, so accelerate with the same a because they are always in contact.

- Considering all three blocks together

$$3F = 3ma$$

$$F = ma$$

- Force on the first block is

$$3F - R_{12} = ma$$

$$R_{12} = 2F$$

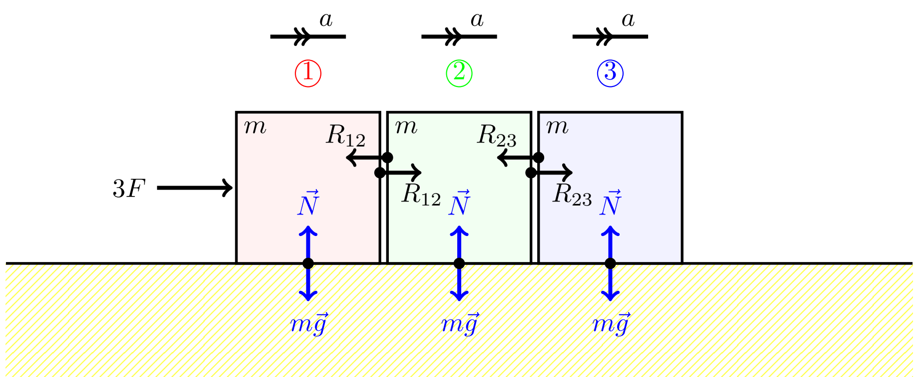

Figure 7.8: Row of three blocks in contact with one another. The left-most block is being pushed by a force $3F$.

- Force on the second block is

$$R_{12} - R_{23} = ma$$
$$2F - R_{23} = ma$$
$$R_{23} = F$$

- Force on the third block is

$$R_{23} = ma$$
$$\text{i.e., } F = ma$$

Note: if we also include friction, the force opposing the motion of each block is $f = \mu N$. Then, considering all three blocks together, we would have

$$3F - 3\mu N = 3ma$$

or for each block individually

$$F - \mu N = a$$

including friction.

7.3.2 Weights and pulleys

Another "standard" problem involved weights suspended from a pulley. This is known as an *Atwood machine*, as shown in Figure 7.9.

When dealing with ropes, the argument is that the tension is the same throughout, assuming the rope has noe weight, and that there are lots of balancing internal forces (in equal and opposite pairs).

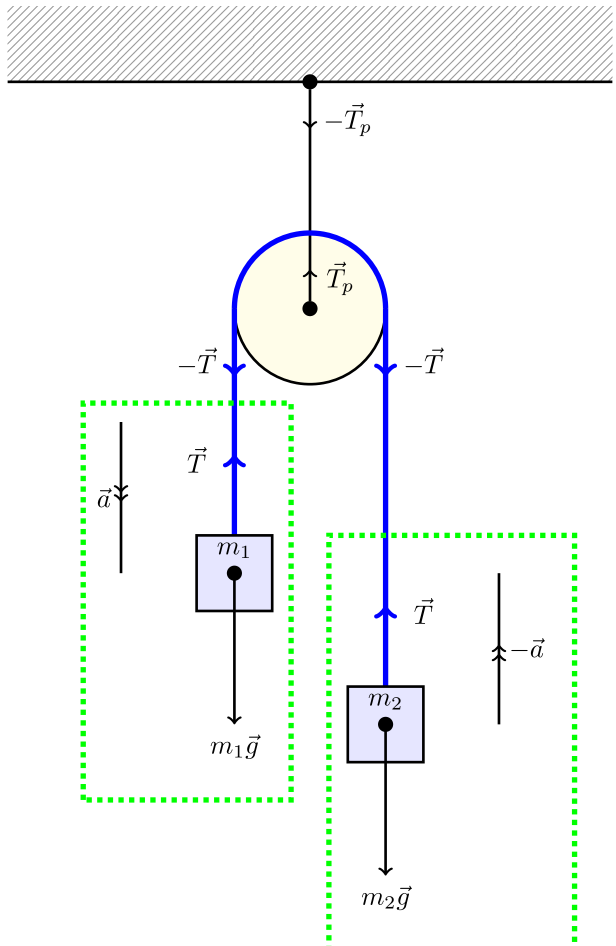

Figure 7.9: Atwood machine

Consider only the forces acting on the left-hand weight,

$$\vec{T} + m_1\vec{g} = m_1\vec{a}$$

Similarly, for the right-hand weight,

$$\vec{T} + m_2\vec{g} = -m_2\vec{a}$$

We can eliminate $\vec{T}$ to get

$$\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{g} \tag{7.1}$$

but $\vec{T} = m_1(\vec{a} - \vec{g})$ so

$$\begin{aligned}\vec{T} &= m_1\left(\frac{m_1 - m_2}{m_1 + m_2} - 1\right)\vec{g} \\ &= -2\left(\frac{m_1 m_2}{m_1 + m_2}\right)\vec{g} \\ \vec{T} &= -2\mu\vec{g}\end{aligned}$$

where, as before, μ is the reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

7.4 Rigid bodies

When we looked at systems of particles in Section 4.2, we found that the momentum of a system of particles can be written as

$$\vec{p}_{\text{tot}} = M\vec{v}_{\text{com}}$$

where M is the total mass of the system, and $\vec{v}_{\text{com}}$ is the velocity of the centre of mass.

- By Newton's first law, for an isolated system (i.e., with no external forces) momentum is conserved.
- By Newton's third law, all particles interact by action-reaction pairs, but this does not affect the centre of mass motion.
- If an external force is applied, it will act on some or all of the particles, and cause a momentum change. We know from Newton's second law that

  $$\text{Total force } \vec{F} = M\frac{d\vec{v}_{\text{com}}}{dt}$$

  i.e., the external force accelerates the system as if acting on total mass, acting at the centre of mass.

A *rigid body* is then defined as an extended object of fixed shape and volume. The above analysis holds, with the constraint that the action-reaction pairs of Newton's third law hold the particles in a rigid arrangement.

Therefore, when considering a rigid body, external forces are taken to act through the centre of mass, and cause acceleration of the total mass.

However, rigid bodies additionally may rotate about some axis. In general a force that does not act through the centre of mass will additionally lead to a rotation (see Chapter 8).

In general, motion of a rigid body will be a combination of

a) motion of the centre of mass
b) rotation about the centre of mass

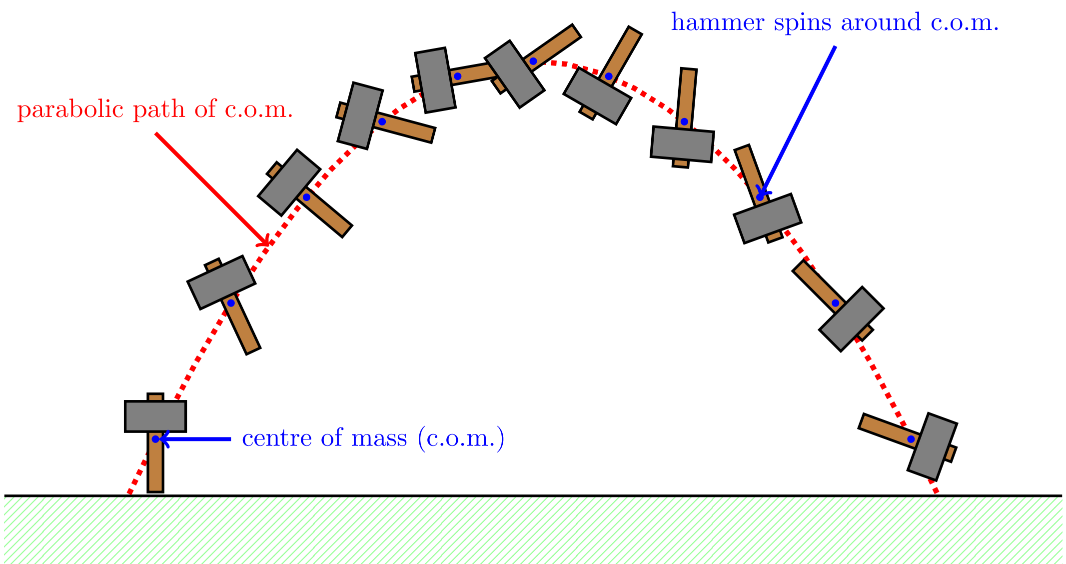

Figure 7.10: Hammer throwing.

8 Rotational mechanics

Before getting into rotational mechanics properly, we'll first introduce the idea of the *cross product* or *vector product* because that will turn out to be particularly useful in our description of rotational motion and angular momentum in particular.

8.1 Cross product

Earlier in the course we looked at the *scalar product* that takes two vectors and produces a scaler. We are now going to introduce something called the *vector* product, or *cross product*, which takes two vectors and produces a third *vector*.

For a geometrical interpretation of the cross product, consider two vectors $\vec{a}$ and $\vec{b}$, with an angle θ between them, as shown in Figure 8.1. The area of the parallelogram with these vectors as sides is $A = |\vec{a}||\vec{b}|\sin\theta$. We can use this to define a vector that is in the direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, and has a magnitude equal to the area of the parallelogram. The direction of this vector is chosen to produce a *right-handed* set of vectors. This is the cross product of $\vec{a}$ and $\vec{b}$, written $\vec{a} \times \vec{b}$.

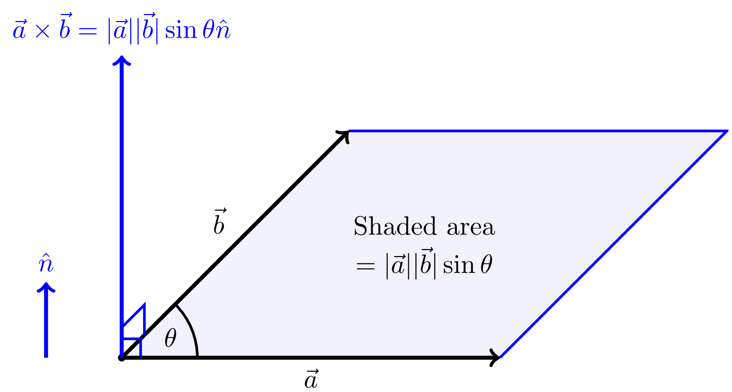

Figure 8.1: Geometrical interpretation of the cross product of two vectors $\vec{a}$ and $\vec{b}$. The cross product $\vec{a} \times \vec{b}$ has a magnitude equal to the area of the parallelogram defined by the two vectors, and is in a direction perpendicular to the plane of those vectors, in a right-handed sense.

Vector product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$$

where θ is the angle between the vectors, and $\hat{n}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$ that forms a *right-handed* set of vectors.
Note: if $\vec{a}$ and $\vec{b}$ are parallel, $\vec{a} \times \vec{b} = \vec{0}$.

Also, because of the need for the right handed set of vectors, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. We say that the vector product is *non-commutative*, i.e., the order of the vectors matters.

We can determine the components of the cross product by considering the cross product of units vectors in a cartesian coordinate system, which is right-handed (see Figure 8.2).

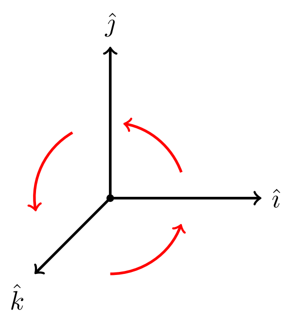

Figure 8.2: Right handed coordinate basis vectors.

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath}$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath}$$

And similarly, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, and $\hat{\imath} \times \hat{k} = -\hat{\jmath}$.

Since the cross product of parallel vectors is zero, we also note that $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$.

The vector product is *distributive*, which means that

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

but it is *not* associative, which means that

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

so when you are writing cross products with three vectors like this the brackets are important. As a concrete example that illustrates this, consider the difference between $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath})$ and $(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath}$.

We can now determine the cross product of arbitrary vectors as follows.

$$\vec{a} \times \vec{b} = (a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \times (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k})$$

which we can multiply out and use the above properties of the basis unit vectors to get

Components of vector product – cyclic permutations

$$\vec{a} \times \vec{b} = (a_yb_z - a_zb_y)\hat{\imath} + (a_zb_x - a_xb_z)\hat{\jmath} + (a_xb_y - a_yb_x)\hat{k}$$

Note the pattern of cyclic permutations. This can also be visualised graphically as shown below.

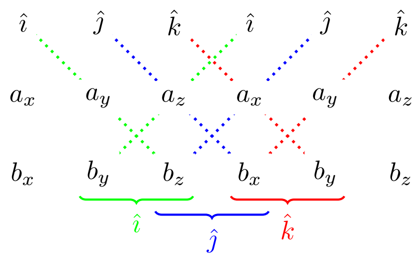

Figure 8.3: Components of vector product

Components of vector product – determinant method

There is another way to calculate the vector product. This is the *determinant method* which is the same as calculating the determinant of a 3×3 matrix. You will find out more about matrices and determinants later in your degree programme, but for now it will be useful just to know this method for calculating the vector product. The technique is illustrated in Figure 8.4.

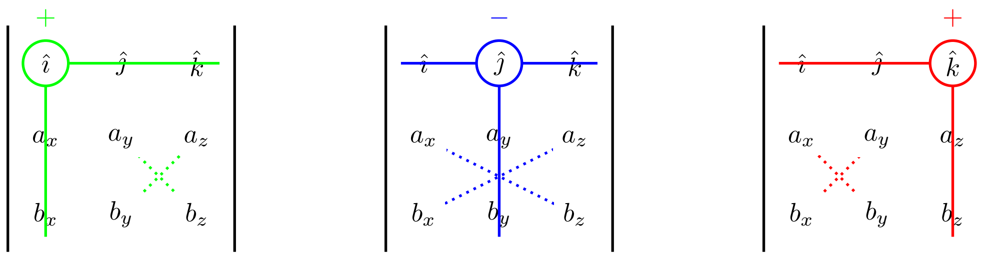

Figure 8.4: Determinant method for calculating the vector product from the components of two vectors $\vec{a}$ and $\vec{b}$. Each circled unit basis vector, with sign above it, is multiplied by the *cofactor* indicated by the dotted lines.

$$\vec{a} \times \vec{b} = +(a_y b_z - b_y a_z)\hat{\imath} - (a_x b_z - b_x a_z)\hat{\jmath} + (a_x b_y - b_x a_y)\hat{k} \tag{8.1}$$

Note the "−" sign associated with the $\hat{\jmath}$ term.

8.2 Circular motion

Circular motion is produced by a constant net force that is perpendicular to a particle's velocity.

8.2.1 Position

If a particle is moving around a circle at constant speed (but not constant velocity because its direction is continuously changing), then we can write its coordinates at time t as:

$$x = r_0 \cos(\omega t)$$

$$y = r_0 \sin(\omega t)$$

where r is the radius of the circle and ω is the angular velocity of the particle. This is illustrated in Figure 8.5, where $\phi = \omega t$.

We can write this as a vector,

$$\vec{r} = r_0(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath})$$

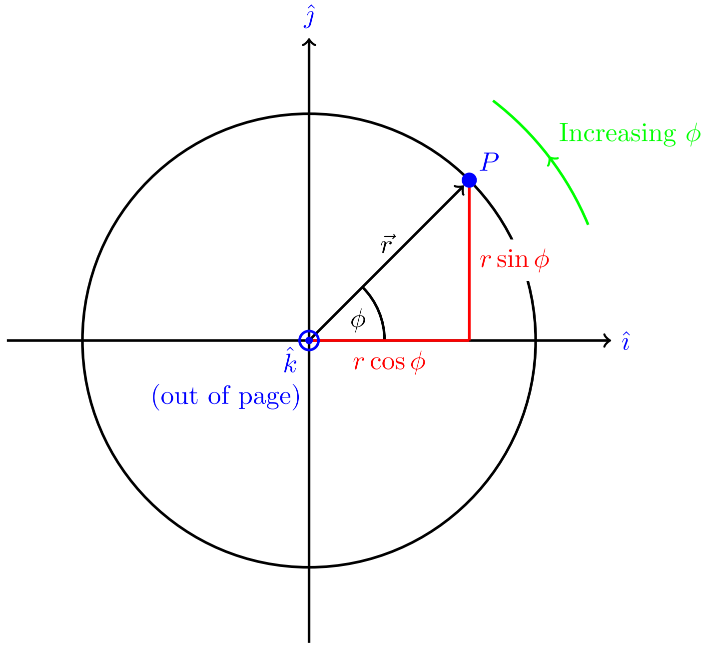

Figure 8.5: Circular motion of particle P. The angle $\phi = \omega t$. The coordinates of P are (x, y) or (r, ϕ).

8.2.2 Velocity

The angular velocity is $\dot{\phi}$, the rate of change of the angle ϕ with time,

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{d}{dt}(\omega t) = \omega$$

The angular velocity ω is related to the period of the motion, T (the time to go once around the circle), by

$$\omega = \frac{2\pi}{T}$$

with units of radians per second.

In terms of frequency, f,

$$\omega = 2\pi f$$

The particle travels a distance equal to the circumference of the circle ($2\pi r_0$) in one period (T), so the speed of the particle is

$$v = \frac{2\pi r_0}{T} = 2\pi r_0 \times \frac{\omega}{2\pi} = r_0\omega$$

Another way to calculate the velocity of the particle is to differentiate its position with respect to time

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -r_0\omega\sin(\omega t)\hat{\imath} + r_0\omega\cos(\omega t)\hat{\jmath}$$

This is a directed tangentially. The speed is given by

$$v^2 = |\vec{v}|^2 = r_0^2\omega^2\underbrace{\left(\sin^2(\omega t) + \cos^2(\omega t)\right)}_{1}$$

So, as before,

$$v = r_0\omega \tag{8.2}$$

Note: $\vec{v} \cdot \vec{r} = 0$, so the velocity and position vectors are perpendicular.

8.2.3 Acceleration

We find the acceleration by differentiating the velocity with respect to time.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -r_0\omega^2\cos(\omega t)\hat{\imath} - r_0\omega^2\sin(\omega t)\hat{\jmath}$$

$$\vec{a}(t) = -r_0\omega^2\underbrace{\left(\cos(\omega t)\hat{\imath} + \sin(\omega g)\hat{\jmath}\right)}_{\text{radial unit vector, }\hat{r}}$$

$$\vec{a}(t) = -r_0\omega^2\hat{r}$$

Hence the centripetal acceleration, a, is given by

$$a = r_0\omega^2 = \frac{v^2}{r_0}$$

The centripetal force is similarly

$$F = mr_0\omega^2 = \frac{mv^2}{r_0}$$

Both the force and acceleration are directed towards the centre of the circle.

$$\vec{F} = -mr_0\omega^2\hat{r} = -\frac{mv^2}{r_0}\hat{r}$$

Note: Since $\vec{v} \cdot \vec{F}_{\text{cent}} = 0$, a centripetal force does no work.

8.3 Angular velocity vector

We can combine the work of the previous two sections to write the angular velocity as a *vector*. This is useful because it tells us the axis of rotation, and will be useful for rotational mechanics in general.

We therefore choose the axis of rotation to define the direction of our angular velocity and use the cross product to define angular velocity vector as follows.

Angular velocity vector

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2} \tag{8.3}$$

This is a vector that is perpendicular to the plane of the particle's motion, and points in the direction of the axis of rotation.

Note: because this is a *vector product* the order of the terms is important to get the direction right.

For circular motion with angular velocity ω,

$$|\vec{v}| = r\omega$$

in a direction perpendicular to $\vec{r}$. This means that

$$\vec{\omega} = \omega\hat{n}$$

where $\hat{n}$ is a unit vector in the direction of the axis of rotation.

8.4 Angular acceleration vector for circular motion

This section *only* applies to circular motion, i.e., when $\hat{\omega}$ is fixed but $|\omega|$ is changing. The *angular acceleration* $\vec{\alpha}$ is the time derivative of $\vec{\omega}$.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d}{dt}\left(\frac{\vec{r} \times \vec{v}}{r^2}\right)$$

For *circular motion* the radius r is constant (i.e., $dr/dt = 0$).

$$= \frac{1}{r^2}\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right)$$

$$= \frac{1}{r^2}\left(\underbrace{\vec{v} \times \vec{v}}_{\vec{0}} + \vec{r} \times \vec{a}\right)$$

Angular acceleration vector for circular motion

$$\vec{\alpha} = \frac{\vec{r} \times \vec{a}}{r^2} \tag{8.4}$$

Note: this *only* applies for *circular motion*.

Note that $\vec{\alpha}$ is zero if the circular motion is at constant speed.

Assume the circular motion is *not* at constant speed. In scalars, we can write $v = r\omega$. Differentiating this gives the acceleration in the tangential direction, $a_t = r\frac{d\omega}{dt}$. We arrive the following expression.

$$r|\vec{\alpha}| = r \times \frac{r \times r\frac{d\omega}{dt}}{r^2} = a_t$$

where a_t is the acceleration in the tangential direction.

8.5 Torque and moment of inertia

Torque is the rotational analogue of force. It is also known as the moment of force.

Torque

Torque is a vector quantity, $\vec{\Gamma}$, defined as follows

$$\vec{\Gamma} = \vec{r} \times \vec{F} \tag{8.5}$$

Where $\vec{r}$ is the vector from the origin to the point at which the force $\vec{F}$ is being applied. It is perpendicular to both $\vec{r}$ and $\vec{F}$, and points in the direction of the axis of rotation. The magnitude of the torque is given by

$$|\vec{\Gamma}| = |\vec{r}||\vec{F}| \sin\theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$. In other words, it is the perpendicular component of the force times the distance from the axis of rotation.

An illustration of torque is shown in Figure 8.6.

The units of torque are newton metres (Nm).

8.5.1 Circular motion

We can find an analogue to Newton's second law by noting that

$$\vec{\Gamma} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{a}$$

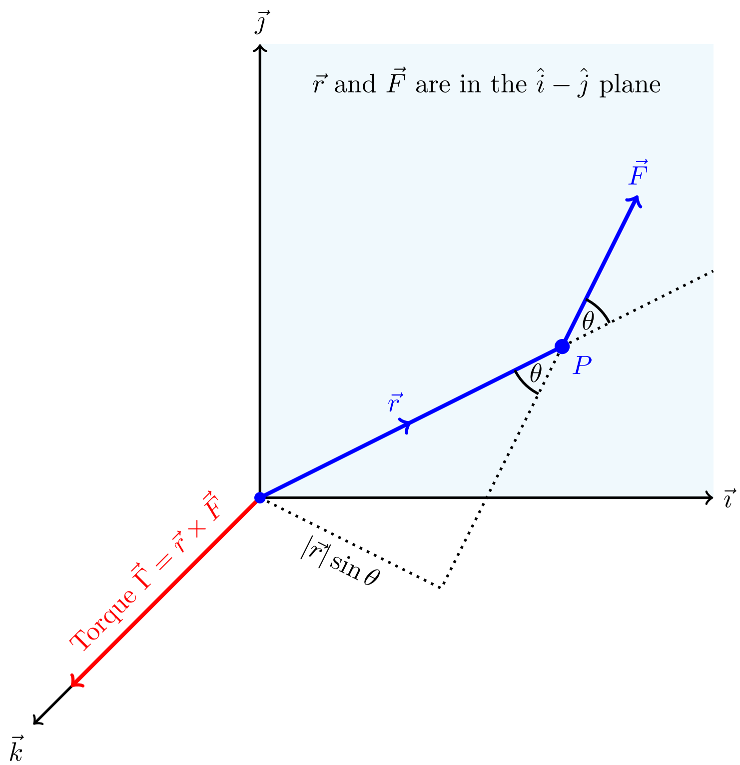

Figure 8.6: Torque produced by a force $\vec{F}$ acting at position $\vec{r}$. In this diagram the vectors $\vec{r}$ and $\vec{F}$ are in the $\hat{\imath}$-$\hat{\jmath}$ plane. The perpendicular distance from the origin to the line of the force is labelled $|\vec{r}| \sin \theta$. The torque in this case is in the direction of the $\hat{k}$ axis and is $\vec{\Gamma} = \vec{r} \times \vec{F}$.

Comparing this with Equation 8.4 we see that *for circular motion* the torque can be written as follows.

Torque and angular acceleration for circular motion

For *circular motion* we find the following relationship between torque, $\vec{\Gamma}$, and angular acceleration, $\vec{\alpha}$,

$$\vec{\Gamma} = mr^2\vec{\alpha} \tag{8.6}$$

where r is the radius of the circle for a point particle of mass m.

This is the rotational analogue to Newton's second law. The torque is proportional to the angular acceleration, but the constant is not m, rather it is mr^2 which implies the distance of the particle from the rotation axis will affect its ability to increase the angular velocity.

8.5.2 Torque on a rigid body

Consider a rigid body made up of N, a large number, of individual masses m_1, m_2, ..., at positions $\vec{r}_1$, $\vec{r}_2$, ..., each acted upon by forces $\vec{F}_1$, $\vec{F}_2$, ... This is a rigid body so the distances between the masses are fixed and there is a single angular velocity vector $\vec{\omega}$. The origin is placed at the centre of rotation, as shown in Figure 8.7, which need not be the centre of mass.

The torque acting on each individual mass is given by

$$\vec{\Gamma}_i = \vec{r}_i \times \vec{F}_i$$

which, looking at Equation 8.6, is equivalent to

$$\vec{\Gamma}_i = m_i r_i^2 \vec{\alpha}$$

where $\vec{\alpha}$ is the same for each mass because this is a rigid body.

The total torque is the sum of each individual torque, noting that for a rigid body $\vec{\alpha}$ is the same for each element,

$$\vec{\Gamma}_{\text{tot}} = \sum_{i=1}^{N} \vec{\Gamma}_i = \sum_{i=1}^{N} m_i r_i^2 \vec{\alpha} = \vec{\alpha} \sum_{i=1}^{N} m_i r_i^2$$

8.5.3 Moment of inertia

This leads us to the following definition of the *moment of inertia* of a rigid body.

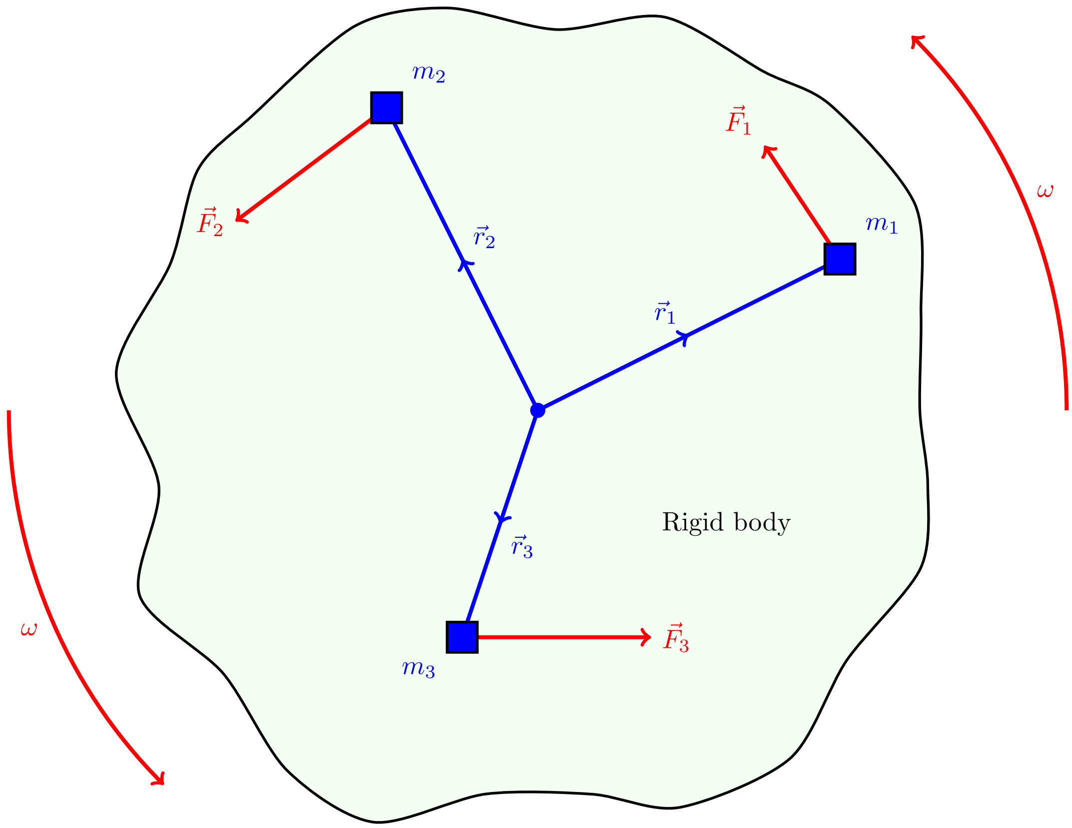

Figure 8.7: Torque on a rigid body made up of N individual masses, m_1, m_2, ... at positions $\vec{r}_1$, $\vec{r}_2$, ... acted upon by forces $\vec{F}_1$, $\vec{F}_2$, ... The origin is placed at the centre of rotation, which need not be the centre of mass. Because this is a rigid body there is one common angular velocity , ω. The figure only shows three of these masses for clarity.

Moment of inertia

The moment of inertia, I, of a rigid body is defined as follows

$$I = \sum_{i=1}^{N} m_i r_i^2 \tag{8.7}$$

where m_i is the mass of the ith element of the body, and r_i is the distance of that element from the axis of rotation.
Moment of inertia has units of kg m^2.

Rotational equivalent of Newton's second law

The total torque $\vec{\Gamma}_{\text{tot}}$ on a rigid body is related to its angular acceleration $\vec{\alpha}$ by

$$\vec{\Gamma}_{\text{tot}} = I\vec{\alpha}$$

where I is the moment of inertia of the rigid body (Equation 8.7).

The role of I in rotational dynamics is analogous to m in linear dynamics, i.e., it resists angular acceleration.

If the torques balance, $\vec{\Gamma} = \vec{0}$ and there is no change in angular velocity.

Example 1: Levers – adding torques

Consider a system that is balanced so there is no net torque about the origin, as shown in Figure 8.8.

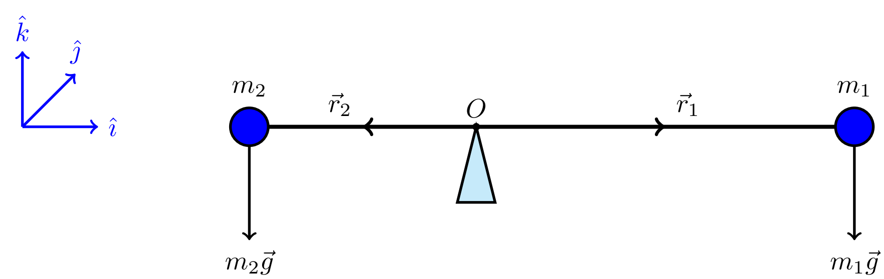

Figure 8.8: A system "balanced" so there is no net torque about the origin.

Using the right handed coordinate system shown in blue in the figure, we can write the radii and forces as follows.

$$\vec{r}_1 = r_1\hat{\imath}$$

$$\vec{r}_2 = -r_2\hat{\imath}$$

$$\vec{F}_1 = m_1\vec{g} = -m_1 g\hat{k}$$

$$\vec{F}_2 = m_2\vec{g} = -m_2 g\hat{k}$$

The torques of the two particles are then obtained by using $\vec{\Gamma}_i = \vec{r}_i \times \vec{F}_i$ as follows

$$\vec{\Gamma}_1 = r_1\hat{\imath} \times (-m_1 g\hat{k}) = -m_1 g r_1(\hat{\imath} \times \hat{k}) = m_1 g r_1\hat{\jmath}$$

$$\vec{\Gamma}_2 = -r_2\hat{\imath} \times (-m_2 g\hat{k}) = m_2 g r_2(\hat{\imath} \times \hat{k}) = -m_2 g r_2\hat{\jmath}$$

If the system is balanced the sum of these moments is zero

$$\vec{\Gamma}_1 + \vec{\Gamma}_2 = \vec{0}$$

$$m_1 g r_1 - m_2 g r_2 = 0$$

$$m_1 r_1 = m_2 r_2$$

This is an example of the principle of moments.

Example 2: Moment of inertia of a ring

Here we calculate the moment of inertia of a ring about an axis perpendicular to the plane of the ring, passing through its centre.

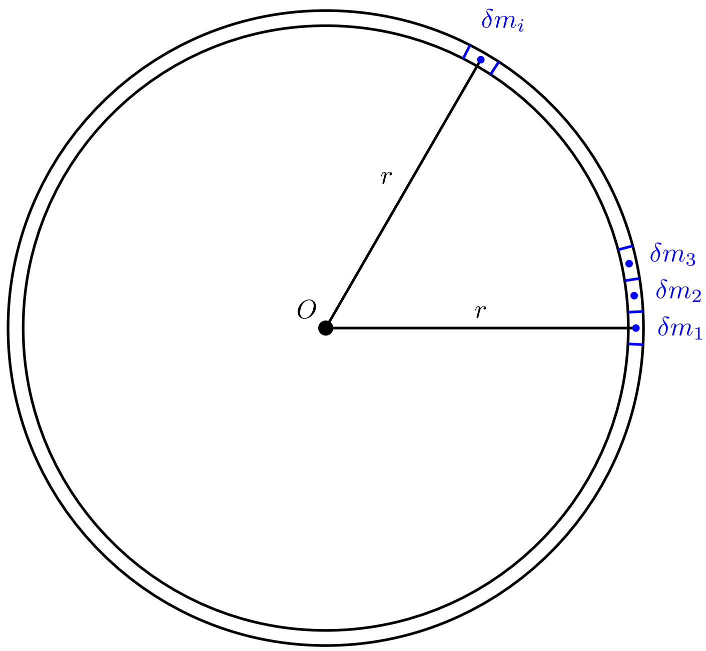

Figure 8.9: Moment of inertia of a circular ring of radius r and total mass M. The ring can be broken up into individual pieces of mass δm_i.

The ring can be thought of as a collection of little masses δm_i at equal distances r from centre, as shown in Figure 8.9. The moment of inertia is given by

$$I = \sum_i \delta m_i r^2$$

But the $\sum_i \delta m_i = M$ where M is the total mass of the ring.
We therefore find that for a ring, the moment of inertia, I, is given by

$$I = Mr^2$$

about an axis through 0 perpendicular to the plane of the ring.

8.6 Angular momentum

By analogy with torque, angular velocity, etc. we define for a particle with linear momentum $\vec{p}$, the *angular momentum* $\vec{L}$ as follows.

Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \tag{8.8}$$

where $\vec{r}$ is the vector from the origin to the particle, and $\vec{p}$ is the linear momentum of the particle. This is illustrated in Figure 8.10.

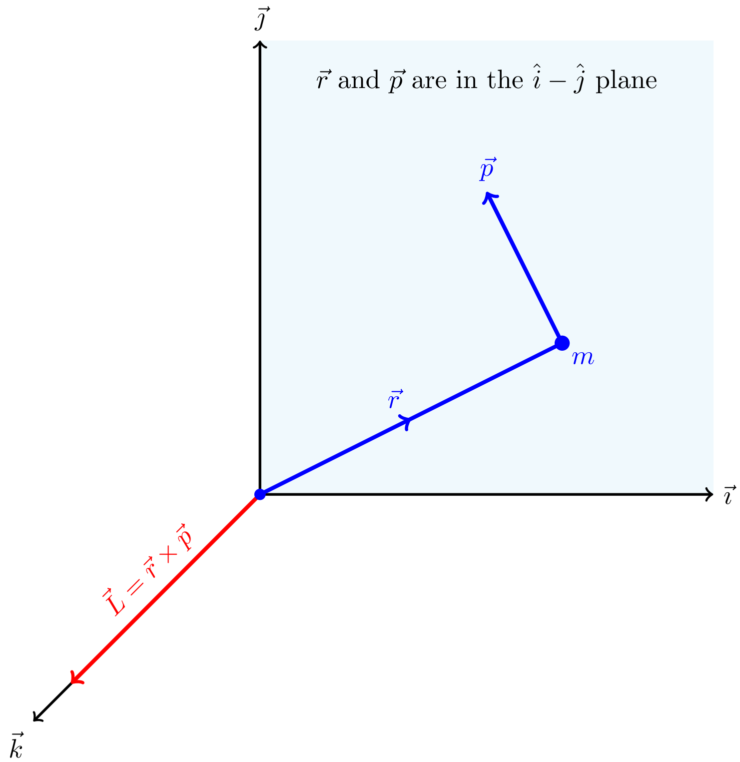

Figure 8.10: The angular momentum is defined to be $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p}$ is the linear momentum.

For *circular motion* $\vec{r}$ and $\vec{v}$ are perpendicular, so $\vec{r}$ and $\vec{p}$ are perpendicular.

$$|\vec{L}| = mvr\sin(90) = mvr = mr^2\omega$$

8.6.1 Angular momentum of a rigid body

Again we can consider splitting the body up into individual elements of mass m_i and momentum $\vec{p}_i$ (similar to what we did for torque, as illustrated by Figure 8.7).

The total angular momentum is then the sum of the angular momenta of the individual elements.

I.e.,

$$\vec{L}_{\text{tot}} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i$$
$$= \sum_i \vec{r}_i \times m_i \vec{v}_i$$

Recall (Equation 8.3) that $\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$, so we can write

$$\vec{L}_{\text{tot}} = \sum_i m_i r_i^2 \vec{\omega}$$

$\vec{\omega}$ is the same for each element because it is a solid body, and $\sum_i m_i r_i^2$ is the moment of inertia, I. Hence we find that

> Angular momentum of a rigid body
>
> $$\vec{L} = I\vec{\omega}$$
>
> where I is the moment of inertia of the rigid body, and $\vec{\omega}$ is the angular velocity of the rigid body.

8.6.2 Conservation of angular momentum

Angular momentum is conserved separately from linear momentum.

Just as force rate of change of linear momentum so torque is the rate of change of angular momentum. Looking at Equation 8.8 ($\vec{L} = \vec{r} \times \vec{p}$), Equation 8.5 ($\vec{\Gamma} = \vec{r} \times \vec{F}$), and considering Newton's second law Equation 5.1 ($\vec{F} = d\vec{p}/dt$), we see that

$$\vec{\Gamma} = \frac{d\vec{L}}{dt}$$

This immediately shows that if there is no net torque on a system then the total angular momentum is conserved.

> Example: mass on rope winding around pole
>
> As a rope wraps around pole its horizontal distance from the pole decreases from r_1 to r_2, as shown in Figure 8.11.

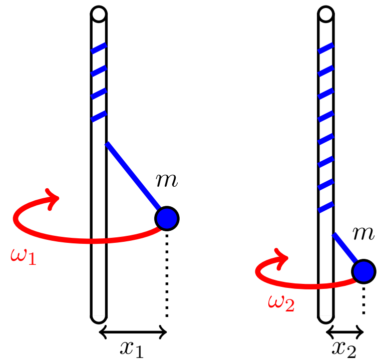

Figure 8.11: As mass m on a rope wraps around a pole its horizontal distance from the pole decreases from r_1 to r_2.

The initial and final moments of inertia of the mass about the pole are

$$I_1 = mr_1^2$$

$$I_2 = mr_2^2$$

The corresponding expressions for angular momentum are

$$L_1 = I_1\omega_1$$

$$L_2 = I_2\omega_2$$

Angular momentum is conserved so $L_1 = L_2$ which implies

$$I_1\omega_1 = I_2\omega_2$$

$$\omega_2 = \frac{I_1}{I_2}\omega_1$$

Since $I_1 > I_2$ we find that $\omega_2 > \omega_1$ and the rotational frequency increases.

8.7 Rotational kinetic energy

Consider a rigid body rotating freely about its centre of mass, shown in Figure 8.12. As before, we can consider splitting the body into N elements of mass m_i, each with momentum $\vec{p}_i$. The total kinetic energy is the sum of the kinetic energies of the individual elements.

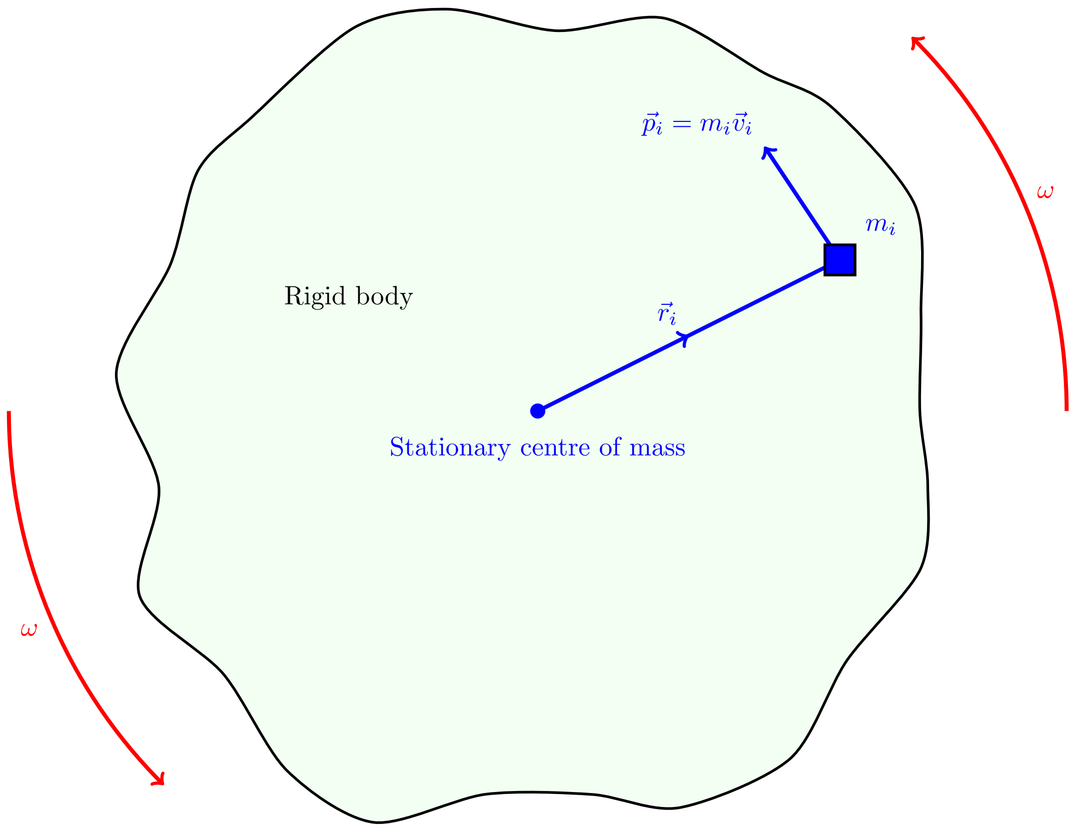

Figure 8.12: Rotational kinetic energy can be found by summing the kinetic energies of N individual elements, each of mass m_i and momentum $\vec{p}_i$.

The total rotational kinetic energy is then

$$\text{KE}_{\text{rot}} = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

But $v_i = r_i \omega$ (Equation 8.2), so

$$\text{KE}_{\text{rot}} = \sum_{i=1}^{N} \frac{1}{2} m_i r_i^2 \omega^2$$

and we have just shown (Equation 8.7) that $I = \sum_i m_i r_i^2$. This gives us an expression for the rotational kinetic energy in terms of the moment of inertia and the angular velocity.

> Rotational kinetic energy
>
> $$\text{KE}_{\text{tot}} = \frac{1}{2} I \omega^2$$
>
> where I is the moment of inertia of the rigid body, and ω is the angular velocity of the rigid body.

A rigid body carries kinetic energy in two forms:

1) linear KE through centre of mass motion and
2) rotational KE about the centre of mass.

> Total kinetic energy of a rigid body
>
> $$\text{KE}_{\text{tot}} = \frac{1}{2} M v_{\text{com}}^2 + \frac{1}{2} I \omega^2$$
>
> where M is the total mass of the rigid body, v_{com} is the velocity of the centre of mass, I is the moment of inertia of the rigid body about the centre of mass, and ω is the angular velocity.

8.8 Conservation of mechanical kinetic energy

When conserving mechanical energy all forms of energy should be considered

- Potential energy
- Linear KE
- Rotational KE
- Linear work
- Rotational work

8.8.1 Rotational work

Consider a force $\vec{F}$ producing a torque $\vec{\Gamma}$ resulting in a small angular change $d\theta$, as shown in Figure 8.13.

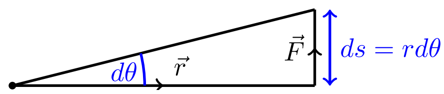

Figure 8.13: Rotational work produced by a torque $\vec{\Gamma}$ due to a force $\vec{F}$ resulting in a small angular change $d\theta$.

For small $d\theta$ the displacement will be $ds = rd\theta$ and the work done will be $dW = Fds = Frd\theta = \Gamma d\theta$.

We can integrate dW to find the work done for a large angular rotation.

Consider constant torque acting to rotate a body from an angle $\theta = \theta_0$ to $\theta = \theta_1$.

The rotational work is equal to the constant torque times the angle moved through.

$$\text{Rotational work} = \int_{\theta_0}^{\theta_1} \Gamma d\theta = \int_{\theta_0}^{\theta_1} I\alpha d\theta$$

.

Using the chain rule we can write

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$$

Therefore

$$\text{Rotational work} = \int_{\theta_0}^{\theta_1} I\omega\frac{d\omega}{d\theta}d\theta = I\int_{\omega_0}^{\omega_1} \omega d\omega = \frac{1}{2}I\omega_1^2 - \frac{1}{2}I\omega_0^2$$

Rotational work

$$W = \underbrace{\frac{1}{2}I\omega_1^2 - \frac{1}{2}I\omega_0^2}_{\text{Change in rotational kinetic energy}}$$

where I is the moment of inertia of the rigid body, ω_0 is the initial angular velocity, and ω_1 is the final angular velocity.
Rotational work = change in rotational kinetic energy.

8.9 Moments of inertia

So far we have written the moment of inertia as a sum

$$I = \sum_{i=1}^{N} m_i r_i^2$$

where m_i is the mass of the i^{th} element of the body, and r_i is the distance of that element from the axis of rotation.

In general we want instead to use integration to calculate the moments of inertia of solid bodies by integrating over their area or volume. We'll do some examples of this in the problems class. For now we'll just give the results.

Table 8.1: Moments of inertia of some common objects.

Body	**Moment of inertia**
Point mass m at a distance r from the axis of rotation	$I = mr^2$
Thin circular loop of radius r and mass m rotating about its center	$I = mr^2$
Thin rod of length L and mass m, perpendicular to the axis of rotation, rotating about its center	$I = \frac{1}{12} mL^2$
Thin, solid disc of radius r and mass m	$I = \frac{1}{2} mr^2$
Thin cylindrical shell with open ends, of radius r and mass m	$I = mr^2$
Solid cylinder of radius r and mass m with rotation axis along centre	$I = \frac{1}{2} mr^2$
Hollow sphere of radius r and mass m	$I = \frac{2}{3} mr^2$
Solid sphere of radius r and mass m	$I = \frac{2}{5} mr^2$

8.10 Comparison of linear and circular motion

Table 8.2: Comparison of linear and circular equations

Linear motion	**Equation**	**Circular motion**	**Equation**
Position	$\vec{r}$	Angle	θ
Velocity	$\vec{v}$	Angular velocity	$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$
Acceleration	$\vec{a}$	Angular acceleration	$\vec{\alpha} = \frac{\vec{r} \times \vec{a}}{r^2}$

Linear motion	Equation	Circular motion	Equation
Force	$\vec{F}$	Torque	$\vec{\Gamma} = \vec{r} \times \vec{F}$
Momentum	$\vec{p}$	Angular momentum	$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$
Mass	m	Moment of inertia	$I = \sum_i m_i r_i^2$
Kinetic energy	$\frac{1}{2}mv^2$	Rotational kinetic energy	$\frac{1}{2}I\omega^2$
Work	$\int \vec{F} \cdot \vec{dr}$	Work	$\int \vec{\Gamma} \cdot d\vec{\theta} = \int \Gamma d\theta$
Newton's second law	$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$	Newton's second law	$\vec{\Gamma} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d\vec{L}}{dt}$
Conservation of linear momentum	Conservation of $\vec{p}$	Conservation of angular momentum	Conservation of $\vec{L}$
Power	$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$	Power	$P = \frac{dW}{dt} = \vec{\Gamma} \cdot \vec{\omega}$

Table 8.3: Equations of motion for linear and circular motion.

Linear motion (constant a)	Circular motion (constant $\vec{\alpha}$)
$\vec{v} = \vec{u} + \vec{a}t$	$\vec{\omega} = \vec{\omega}_0 + \vec{\alpha}t$
$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$	$\Delta\theta\hat{k} = \vec{\omega}_0 t + \frac{1}{2}\vec{\alpha}t^2$ (motion in x–y plane)
$v^2 = u^2 + 2\vec{a} \cdot \vec{s}$	$\omega^2 = \omega_0^2 + 2\vec{\alpha} \cdot \vec{k}\Delta\theta$ (motion in x–y plane)

8.11 Examples

It will be helpful to look at some examples of rotational motion to consolidate the ideas introduced in this chapter. There will also be relevant examples in the problems class.

i Example: Atwood machine with massive pulley

Let's revisit the Atwood machine but now with a pulley that has some mass, and friction so that the rope does not slide over the pulley but causes it to rotate. We assume there is sufficient friction to stop the rope slipping over the pulley. The moment of inertia of the pulley is $I = \beta M r^2$ where β depends on the exact nature of the pulley (e.g., $\beta = 1/2$ if the pulley is a solid disc). This is shown in Figure 8.14.

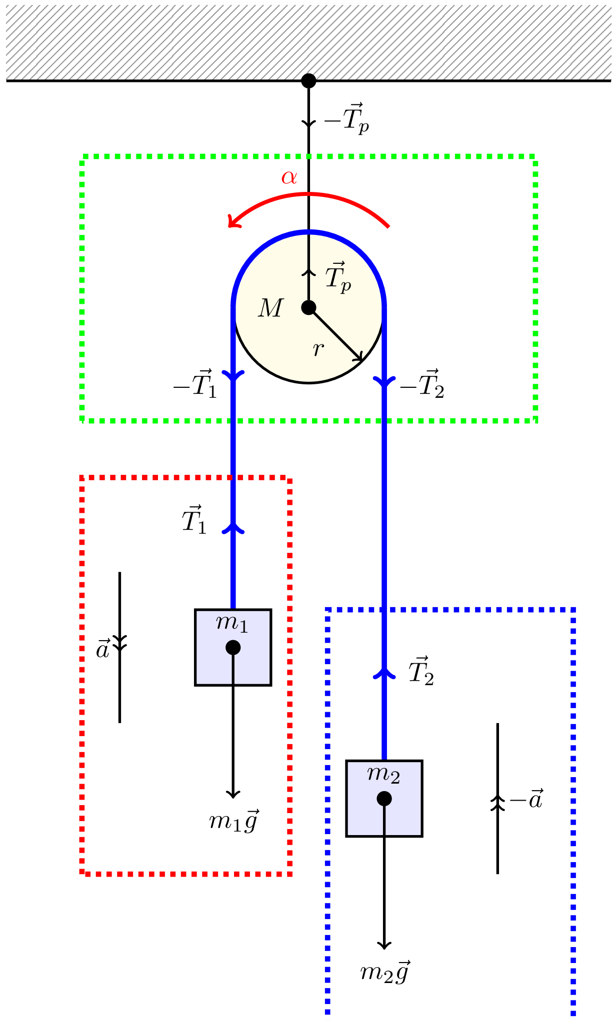

Figure 8.14: An Atwood machine with a massive pulley.

The angular acceleration of the pulley, α, is related to the linear acceleration $|\vec{a}|$ by

$$\alpha = \frac{|\vec{a}|}{r}$$

because the rope does not slip.
This friction also means that the tension is not the same on either side of the pulley, i.e., $|\vec{T}_1| \neq |\vec{T}_2|$.
Let's consider each part (outlined by dotted lines in Figure 8.14) separately.
Left hand weight (outlined in red):

$$m_1 g - T_1 = m_1 a \qquad \text{(downwards)} \tag{8.9}$$

Right hand weight (outlined in blue):

$$T_2 - m_2 g = m_2 a \qquad \text{(upwards)} \tag{8.10}$$

Pulley (outlined in green): Torque about the axis of the pulley (out of the page), anticlockwise is as follows

$$T_1 r - T_2 r = I\alpha = \beta M r^2 \frac{a}{r}$$

$$T_1 - T_2 = \beta M a$$

From Equation 8.9 and Equation 8.10 we can write

$$T_1 = m_1(a - g)$$

$$T_2 = m_2(a + g)$$

Therefore

$$m_1(a - g) - m_2(a + g) = \beta M a$$

which rearranges to give

$$a = \frac{g(m_1 - m_2)}{m_1 + m_2 + \beta M}$$

This agrees with our previous result when $M = 0$ (see Equation 7.1).
$M > 0$ so in general acceleration *lower* than in case non massive pulley.

Cylinder rolling down slope without slipping

Let's consider a cylinder (or ball), with moment of inertia $I = \beta m r^2$, rolling down a slope without slipping, as shown in Figure 8.15.

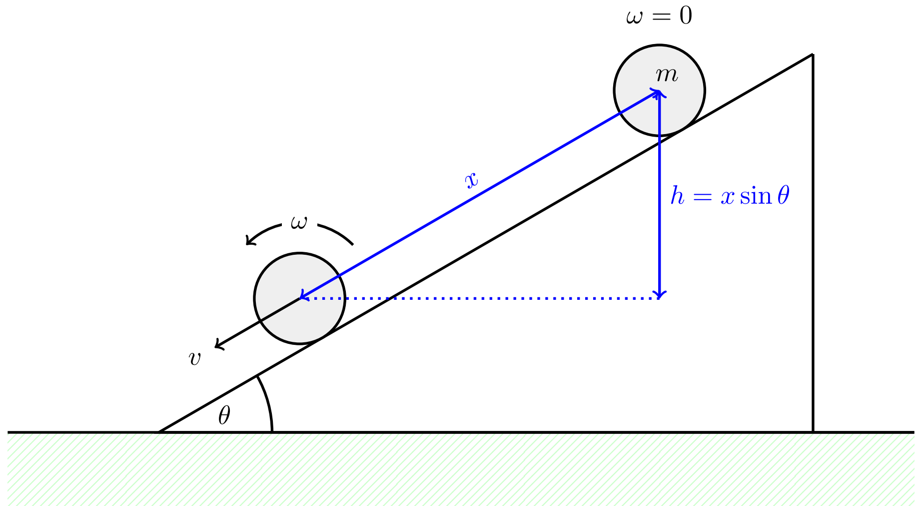

Figure 8.15: A cylinder rolling down a slope without slipping.

We assume the cylinder is not slipping so

$$v = r\omega$$

and no work is done by friction.
We can consider conservation of energy to say

$$\text{PE lost} = \text{linear KE gained} + \text{rotational KE gained}$$

$$mgx\sin\theta = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgx\sin\theta = \frac{1}{2}mv^2 + \frac{1}{2}\beta mr^2\left(\frac{v}{r}\right)^2$$

$$v = \sqrt{\frac{2gx\sin\theta}{1+\beta}}$$

The final speed depends on β and is larger for a sphere ($\beta = 2/5$) than a cylinder ($\beta = 1/2$).
Note: there is no dependence on mass only on β, i.e., shape.

References

Goldstein, Herbert, John Safko, Charles P. (Charles Patton) Poole Jr, Charles P. Poole, and John L. Safko. 2014. *Classical Mechanics.* Third edition, Pearson new international edition. Essex, England: Pearson.

Kleppner, Daniel, and Robert J. Kolenkow. 2014. *An Introduction to Mechanics.* 2nd ed. Cambridge, UK: Cambridge University Press.

Tipler, Paul Allen, Gene Mosca, and Gene P. Mosca. 2008. *Physics for Scientists and Engineers: With Modern Physics.* Sixth edition. New York, NY ; W.H. Freeman.

Part II

Problem classes

9 Problems 1

Week 2

These problems sheets are designed to help you learn mechanics. You get better at physics by solving problems. You will attempt these questions at the problems class. This means you do not need to complete the problems beforehand, but it might be helpful to have looked over them. Your solutions will not be marked. Solutions will be provided on Blackboard the week following your problems class. If you are using these for revision try to do the questions first without looking at the answers.

9.1 Projectile motion

A particle is launched with speed u at an angle θ to the horizontal. The particle experiences a constant acceleration in the vertical direction due to gravity, g.

(a) What are the x and y components of the particle's initial velocity?
(b) What is the maximum height of the projectile?
(c) How long does it take the particle to return to the ground?
(d) What is the range of the particle?
(e) Differentiate your answer to part (d) to show that the maximum range of the particle is achieved when $\theta = 45°$.

Answer to (a)

The x and y components of the particle's initial velocity are

$$u_x = u\cos\theta$$

$$u_y = u\sin\theta$$

Answer to (b)

We only need to consider the vertical motion. There is constant acceleration so we can use the "SUVAT" equations. We can use the "$v^2 = u^2 + 2as$" " equation. The final velocity is zero at the maximum height, so

$$0 = u_y^2 + 2(-g)s$$

$$s = \frac{(u \sin \theta)^2}{2g}$$

Answer to (c)

The particle returns to the ground when its vertical displacement is zero. We can use the "$s = ut + \frac{1}{2}at^2$" equation.

$$s = u_y t - \frac{1}{2}gt^2$$

$$0 = u \sin \theta t - \frac{1}{2}gt^2$$

$$t = 0 \text{ or } t = \frac{2u \sin \theta}{g}$$

Answer to (d)

The range is the horizontal displacement of the particle. We can use the "$s = ut + \frac{1}{2}at^2$" equation with the time we have calculated above. There is no horizontal acceleration.

$$R = u_x t$$

$$= u \cos \theta \times \frac{2u \sin \theta}{g}$$

$$= \frac{2u^2 \sin \theta \cos \theta}{g}$$

$$= \frac{u^2 \sin 2\theta}{g}$$

where we have made use of the double angle formula for $\sin 2\theta = 2 \sin \theta \cos \theta$.

Answer to (e)

We have calculated the range as a function of inclination angle, $R(\theta)$. We want to find the value of θ that maximises the range. We do this by differentiating $R(\theta)$ with respect to θ, and setting this to zero.

$$\frac{dR(\theta)}{d\theta} = \frac{d}{d\theta}\left(\frac{u^2 \sin 2\theta}{g}\right)$$

$$= \frac{u^2}{g} 2 \cos 2\theta = 0$$

$\cos 2\theta = 0$ when $2\theta = 90^\circ$, i.e. when $\theta = 45^\circ$.

9.2 Vector addition

Vectors whose moduli are 3, 4, and 6 act in directions making angles 30°, 90°, and 135° respectively with the positive x-axis, and all lie in the $x - y$ plane. Find their sum. Give the modulus and direction of the resultant vector.

Hint

- One way to do this is to work out the $\vec{\imath}$ and $\vec{\jmath}$ components of each vector, then add them together.
- "Modulus" is another word for "magnitude".

Answer

We can find the components of each vector parallel to the x and y-axes.

$$\vec{v}_1 = 3(\cos 30^\circ \hat{\imath} + \sin 30^\circ \hat{\jmath}) = 2.598\hat{\imath} + 1.5\hat{\jmath}$$

$$\vec{v}_2 = 4(\cos 90^\circ \hat{\imath} + \sin 90^\circ \hat{\jmath}) = 0\hat{\imath} + 4\hat{\jmath}$$

$$\vec{v}_3 = 6(\cos 135^\circ \hat{\imath} + \sin 135^\circ \hat{\jmath}) = -4.243\hat{\imath} + 4.243\hat{\jmath}$$

Sum these to get the resultant vector

$$\vec{R} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = -1.645\hat{\imath} + 9.743\hat{\jmath}$$

The modulus is given by

$$|\vec{R}| = \sqrt{(-1.645)^2 + (9.743)^2} = 9.88 \text{ N}$$

and the direction is given by

$$\theta = \tan^{-1}\left(\frac{9.743}{-1.645}\right) = 99.6^\circ$$

9.3 Angles between vectors

The resultant, $\vec{R}$, of two intersecting forces $\vec{F}_a$ and $\vec{F}_b$ has magnitude $\sqrt{3}$ N. If $|\vec{F}_a| = 1$ N and $|\vec{F}_b| = 2$ N, find the angle between $\vec{F}_a$ and $\vec{F}_b$, and the angle between $\vec{F}_a$ and $\vec{R}$.

Hint

- Draw a diagram.
- Use the cosine rule, and the sine rule.

Answer

Use the cosine rule to find the angle between $\vec{F}_a$ and $\vec{F}_b$

$$|\vec{R}^2| = |\vec{F}_a|^2 + |\vec{F}_b|^2 - 2|\vec{F}_a||\vec{F}_b|\cos\theta$$

i.e.,

$$3 = 1 + 4 - 4\cos\theta$$

and so $\theta = 60^\circ$. From your diagram you want the exterior angle which is 120°. For the angle ϕ between $\vec{F}_a$ and $\vec{R}$ use the sine rule

$$\frac{2}{\sin\phi} = \frac{\sqrt{3}}{\sin 60^\circ}$$

Hence $\sin\phi = 1$ and $\phi = 90^\circ$.

9.4 Non-uniform acceleration with vectors & derivatives

A particle of mass $m = 2$ kg is acted on by a force $\vec{F}$ in newtons. The position $\vec{r}$ of the particle is found to follow a path due to this force given by

$$\vec{r} = \left(\frac{5t^2}{3b} - \frac{t^3}{4c}\right)\hat{\imath} + \left(\frac{3t^2}{b} - \frac{7t}{a}\right)\hat{\jmath}\ \mathrm{m}$$

where a, b and c are constants with the value 1 in appropriate units, when distance is measured in metres and time in seconds. Find the following.

(a) The units of the constants a, b and c.
(b) The value of t when the particle is moving parallel to the vector $\hat{\imath}$.
(c) The force $\vec{F} = m\vec{a}$, where $\vec{a}$ is the acceleration, after 5 seconds.
(d) The magnitude of the force determined in part (c).

Answer to (a)

The units of the constants a, b and c are found by considering the units of the position $\vec{r}$ and time t.
The position $\vec{r}$ has units of metres, and time t has units of seconds. Therefore, the units of the constants a, b and c are:

$$\text{Units of } b = \text{ m}^{-1}\text{ s}^2$$

$$\text{Units of } c = \text{ m}^{-1}\text{ s}^3$$

$$\text{Units of } a = \text{ m}^{-1}\text{ s}$$

Answer to (b)

To get the velocity we need to differentiate $\vec{r}$ with respect to time,

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{10t}{3b} - \frac{3t^2}{4c}\right)\hat{\imath} + \left(\frac{6t}{b} - \frac{7}{a}\right)\hat{\jmath}\text{ m s}^{-1}$$

$\vec{v}$ will be parallel to $\hat{\imath}$ when the $\hat{\jmath}$ component is zero. This occurs when

$$\frac{6t}{b} - \frac{7}{a} = 0$$

$$t = 1.17\text{ s}$$

Answer to (c)

The acceleration is the derivative of the velocity with respect to time,

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \left(\frac{10}{3b} - \frac{6t}{4c}\right)\hat{\imath} + \left(\frac{6}{b}\right)\hat{\jmath}\text{ m s}^{-2}$$

The acceleration after 5 seconds is

$$\vec{a}(t=5) = \left(\frac{10}{3} - \frac{30}{4}\right)\hat{\imath} + (6)\,\hat{\jmath} = -4.17\hat{\imath} + 6\hat{\jmath}\text{ m s}^{-2}$$

The force after 5 seconds is

$$\vec{F}(t=5) = m\vec{a}(t=5) = -8.33\hat{\imath} + 12\hat{\jmath}\text{ N}$$

Answer to (d)

Use Pythagoras' theorem to find the magnitude of the force,

$$|\vec{F}| = \sqrt{(-8.33)^2 + (12)^2} = 14.6 \text{ N}$$

9.5 Centre of mass of system of particles in 3D

Particles 1,2,3 with mass $m_1 = 3$ kg, $m_2 = 6$ kg and $m_3 = 7$ kg are located at positions defined by the vectors $\vec{r}_1 = 5\hat{\imath} - 7\hat{\jmath}$ m, $\vec{r}_2 = 3\hat{\imath} + 6\hat{\jmath}$ m and $\vec{r}_3 = 7\hat{\imath} + 3\hat{\jmath}$ m.

(a) Determine the position of the centre of mass of the three-particle system.
(b) The particles are subsequently displaced in the direction of $\hat{k}$ so that their new positions are given by $\vec{r}_1' = 5\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$ m, $\vec{r}_2' = 3\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$ m and $\vec{r}_3' = 7\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$ m. Determine the new position of the centre of mass of the particle system.
(c) If this translation occurs uniformly over a time t, what is the velocity of the centre of mass frame?

Answer to (a)

Here we use

$$\sum_{i=1}^{3} m_i \vec{r}_i = \vec{r}_{\text{com}} \sum_{i=1}^{3} m_i$$

$$\vec{r}_{\text{com}} = \frac{\sum_{i=1}^{3} m_i \vec{r}_i}{\sum_{i=1}^{3} m_i}$$

$$= \frac{3 \times (5\hat{\imath} - 7\hat{\jmath}) + 6 \times (3\hat{\imath} + 6\hat{\jmath}) + 7 \times (7\hat{\imath} + 3\hat{\jmath})}{3 + 6 + 7} \text{ m}$$

$$= \frac{41}{8}\hat{\imath} + \frac{9}{4}\hat{\jmath} \text{ m}$$

Answer to (b)

The $\hat{\imath}$ and $\hat{\jmath}$ components are unaffected so that the $\hat{k}$ component becomes $(3 \times 3 + 6 \times 6 - 7 \times 3)/16 = 24/16 = 3/2$. Hence the new centre of mass is

$$\vec{r'}_{\text{com}} = \frac{41}{8}\hat{\imath} + \frac{9}{4}\hat{\jmath} + \frac{3}{2}\hat{k} \text{ m}$$

Answer to (c)

If we assume the displacement takes place uniformly over a time t, then the velocity of the centre of mass is

$$\vec{v}_{\text{com}} = \frac{\vec{r'}_{\text{com}} - \vec{r}_{\text{com}}}{t}$$

$$= \frac{3}{2t}\hat{k} \text{ m s}^{-1}$$

9.6 Bonus question: Galilean transformation using vectors

Galileo reputedly tested his ideas about Galilean relativity and gravity by dropping objects from the leaning tower of Pisa, which has height 56 m.

(a) Assume Galileo dropped a book of mass 2.4 kg from the tower. Write down a vector equation for the book's position as a function of time *in Galileo's frame*, i.e. take Galileo to be the origin, and a gravitational force proportional to g in the $-\hat{k}$ direction, and ignore the effects of wind resistance.

(b) A bird was flying near Galileo as he dropped the book, with instantaneous position $3.1\hat{\imath} + 2.0\hat{k}$ m and velocity $1.2\hat{\imath} + 0.6\hat{\jmath}$ m s^{-1} in Galileo's frame. Write down the time-dependent position of the book in the bird's frame. What is the position vector, in the bird's frame, that the book hits the ground?

Hint for part (b)

You might find this much easier to solve if you draw a diagram with vectors showing the positions of the book and the bird relative to Galileo at time t.

Answer to (a)

Using standard "SUVAT" equation for constant acceleration downwards from origin at top of tower

$$\vec{r} = -\frac{1}{2}gt^2\hat{k}$$

Answer to (b)

The bird is initially at $\vec{r}_0 = 3.1\hat{\imath} + 2.0\hat{k}$ m in Galileo's frame and has velocity $\vec{v} = 1.2\hat{\imath} + 0.6\hat{\jmath}$ m s^{-1} relative to Galileo. This is most easily visualised with Figure 9.1.

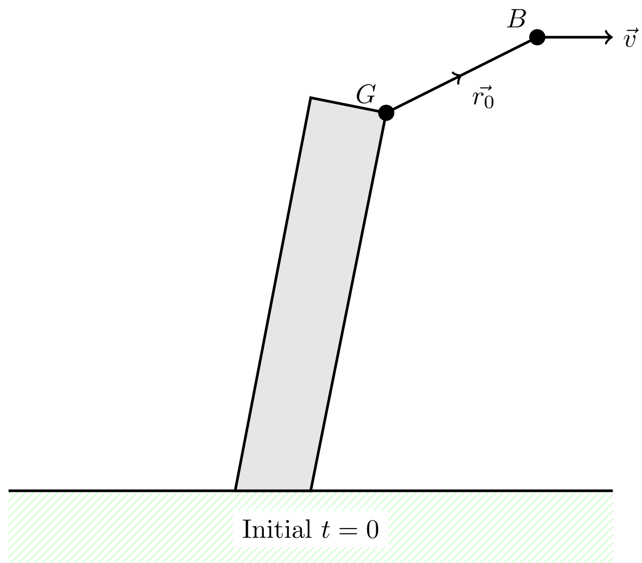

Figure 9.1: Initial condition at $t = 0$, with Galileo, G, at the top of the tower and a bird, B, flying by with velocity $\vec{v}$. At this point Gelileo drops the book.

At some later time $t > 0$ the book is at $\vec{r}(t)$ in Galileo's frame and $\vec{r'}(t)$ in the bird's frame.
At time t the bird is at $\vec{r}_0 + \vec{v}t$ relative to Galileo's frame. This is shown in Figure 9.2.

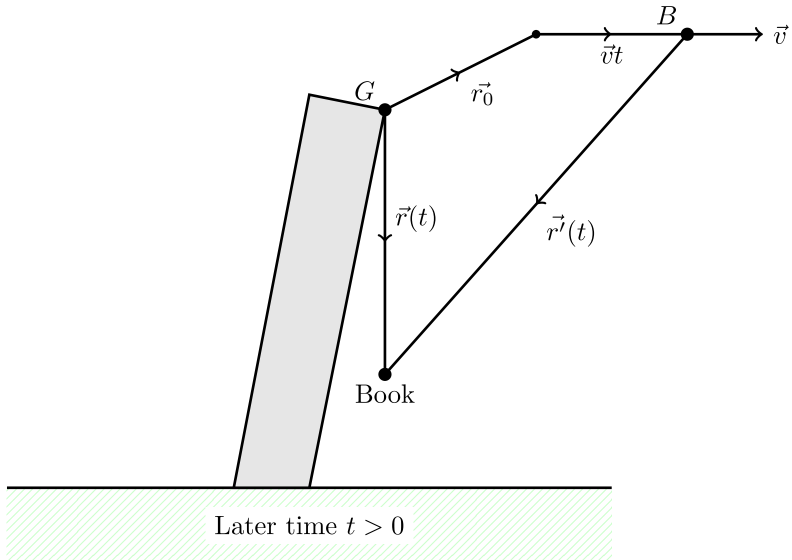

Figure 9.2: At some later time $t > 0$, the book is falling vertically downwards. The bird, B, has flown through a displacement $\vec{v}t$.

From the diagram we can see that the book is at

$$\vec{r'}(t) = \vec{r}(t) - \vec{r}_0 - \vec{v}t$$

relative to the bird. Substituting in the expression for $\vec{r}(t)$ from part (a) gives

$$\vec{r'}(t) = -(3.1 + 1.2t)\hat{\imath} - 0.6t\hat{\jmath} - \left(2.0 + \frac{gt^2}{2}\right)\hat{k}$$

If the tower is 56 m high, the time to fall is 3.4 seconds, giving

$$\vec{r'}(t = 3.4) = -7.2\hat{\imath} - 2.0\hat{\jmath} - 58\hat{k} \text{ m}$$

for the landing position in the bird's frame.

10 Problems 2

Week 4

Problems class questions will be displayed during the class. You do not need to do these in advance of the problems class. The questions will be published the day before the problems class so you can take a look at them if you need to.

10.1 Sliding drink along a bar

The barman in a local nightclub prides himself in being able to slide any drink along the bar so that it comes to rest exactly opposite the correct customer. The coefficient of kinetic friction of the bottom of a glass on the bar is 0.25. If he slides a cocktail to a customer a distance of 3 m along the bar, at what speed must he release the glass from his hand?

Answer

The vertical forces balance in this problem. The friction force acts horizontally and results in the deceleration of the particle (glass containing drink).
In order to work out the friction force, we need the normal reaction which is equal to the particle's weight. Once we know the acceleration, the relationship between the initial speed and the distance travelled can found using SUVAT.
The normal reaction is $\mu_k mg$ so the acceleration is therefore $-\mu_k g$. The distance travelled before coming to rest is found from

$$u^2 + 2as = 0$$

for an initial speed u. So we have

$$u = \sqrt{2\mu_k gs} = \sqrt{2 \times 0.25 \times 9.81 \times 3.0} \text{ m s}^{-1} = 3.8 \text{ m s}^{-1}$$

10.2 Tensions in climbing ropes

A climber of mass 65 kg is in equilibrium with their feet against a rock surface inclined at $\theta = 40°$ to the horizontal, and is supported by a rope running parallel to the rock face. Draw a diagram showing the forces acting on the climber. Find the tension in the rope. (Assume the force of the rock on the climber is at right angles to the surface.)

Answer

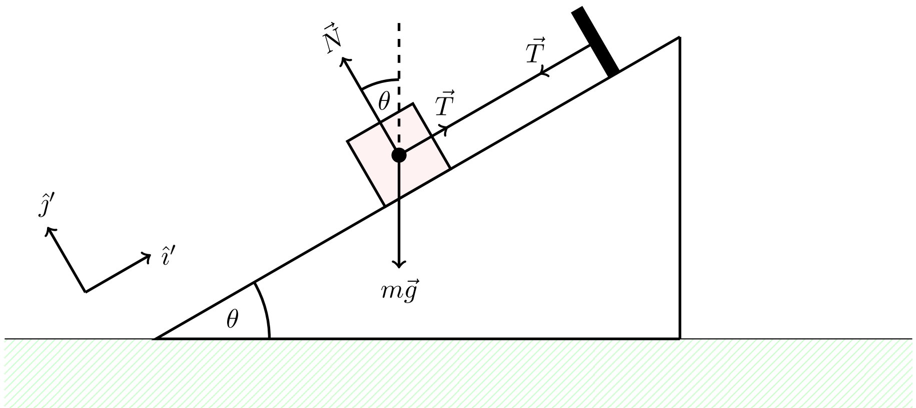

Figure 10.1: Rock climber tethered by a rope. This is the same figure that appeared in the lecture notes. The angle $\theta = 40°$.

This is a case where a little thought, and careful choice of axes, can save a lot of work. Consider Figure 10.1 and resolve the forces parallel and perpendicular to the incline. Parallel to $\hat{\imath}'$ we have

$$T - mg\sin\theta = 0$$

Parallel to $\hat{\jmath}'$ we have

$$N - mg\cos\theta = 0$$

We only need the first of these expressions to determine the tension in the rope, T,

$$T = mg\sin(40) = 410 \text{ N}$$

Note: you can also solve this problem resolving the forces horizontally and vertically. However, resolving parallel and perpendicular to the plane is easier in this case.

10.3 Work done in a lift

A lecturer takes the main lift from the ground to the fourth floor in the Physics building. The lift exerts a force on the lecturer of 800 N for the first 0.45 m of travel. For the next 17.1 m the force is equal to the lecturer's weight, 772 N and for the last 0.45 m it is 744 N. Find:

(a) The total work done by the lift.
(b) The total work done by gravity.
(c) The final kinetic energy of the lecturer.

Answer to part (a)

The work done by the lift is $800 \times 0.45 = 360$ J during the first 0.45 metres; 13.2 kJ over the next 17.1 metres and 334.8 J during the last 0.45 metres. Total work done is 13896 J.

Answer to part (b)

The weight of the lecturer is $mg = 772$ N. The work done by gravity is -772×0.45 J $=$ -347.4 J during the first and last sections of the journey, and -13.2 kJ in the middle section. Total work done is -13896 J.

Answer to part (c)

So the total work done is 12.6 J in the first section, zero in the middle and -12.6 J in the last section. Overall the total work done is zero, so the final kinetic energy of the lecturer must also be zero.

10.4 Work done by force not parallel to displacement

A 2 kg object is given a displacement $\vec{s} = 3\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ m along a straight line. During the displacement, a constant force $\vec{F} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ N acts on the object.

(a) Find the work done as the particle experiences this displacement.
(b) Find the component of $\vec{F}$ in the direction of the displacement.

Answer to part (a)

To find the work W

$$\begin{aligned} W &= \vec{F} \cdot \vec{s} \\ &= (2\hat{\imath} + 1\hat{\jmath} + 1\hat{k}) \cdot (3\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) \text{ N m} \end{aligned}$$

$$= 6 + 3 + 2 \text{ J} = 11 \text{ J}$$

Answer to part (b)

To find the component of $\vec{F}$ in the direction of $\vec{s}$, $F_{\parallel}$, we use the fact that

$$\vec{F} \cdot \vec{s} = |\vec{F}||\vec{s}| \cos\theta$$

We want $F_{\parallel} = |\vec{F}| \cos\theta$, which is

$$F_{\parallel} = \frac{|\vec{F}||\vec{s}| \cos\theta}{|\vec{s}|} = \frac{\vec{F} \cdot \vec{s}}{|\vec{s}|} = \frac{W}{|\vec{s}|}$$

$$|\vec{s}|^2 = 3^2 + 3^2 + 2^2 \text{ m}^2 = 22 \text{ m}^2$$

$$F_{\parallel} = \frac{11}{\sqrt{22}} \text{ N}$$

10.5 Heavy blocks colliding in 1D

A 2.0 kg block moving to the right with a speed of 5.0 m s^{-1} collides with a 3.0 kg block that is moving in the same direction at 2.0 m s^{-1}. After the collision, the 3.0 kg block moves to the right at 4.2 m s^{-1}. Find

(a) the velocity of the 2.0 kg block after the collision.
(b) the coefficient of restitution between the two blocks.

Answer to part (a)

We use momentum conservation here. Since it is a one-dimensional problem and we are given the final speed of one of the particles, as well as the initial conditions, we can solve the problem.
Initial momentum of the system is

$$(2 \times 5 + 3 \times 2) \text{ kg m s}^{-1} = 16 \text{ kg m s}^{-1}$$

The final momentum of the 3 kg block is

$$(3 \times 4.2) \text{ kg m s}^{-1} = 12.6 \text{ kg m s}^{-1}$$

So the final speed of the 2 kg block is

$$\frac{3.4}{2} = 1.7 \text{ m s}^{-1}$$

This result is positive so the block is still moving to the right after the collision.

Answer to part (b)

To find the coefficient of restitution we look at the relative speed before and after the collision. Before the collision it is

$$(5-2)\ \mathrm{m\ s^{-1}} = 3\ \mathrm{m\ s^{-1}}$$

Afterwards the relative speed is

$$(4.2-1.7)\ \mathrm{m\ s^{-1}} = 2.5\ \mathrm{m\ s^{-1}}$$

So the coefficient of restitution is

$$e = \frac{2.5}{3} = 0.83$$

10.6 Bonus question: Sailing boat in light wind

A sailing boat is oriented in a direction $2\hat{\imath} + 3\hat{\jmath}$ on the surface of a lake, and its sail experiences a force from the wind of $-50\hat{\imath} + 250\hat{\jmath}$ N. You can assume the boat experiences no drag in the forward-aft direction, and that the keel provides sufficient resistive force against the water that any net motion of the boat is forwards. You can also assume the boat remains upright.

(a) Draw a free body diagram of the forces acting on the boat. What is the direction of any horizontal force the boat exerts on the water?
(b) Assuming the boat has mass 43 kg and the sailor has mass 78 kg, what is the magnitude of the acceleration of the boat?

Answer to part (a)

A sensible diagram will have the force of the wind $\vec{F}_w$ and the resistance of the boat. In the absence of any other drag, the resistance force of the boat should only point perpendicular to the direction of travel of the boat, and will oppose the component of Fw perpendicular to the keel i.e. $\vec{F}_k$. The resultant force on the boat is parallel to $\hat{n}$ i.e. $\vec{F}_b$. This is shown in Figure 10.2.

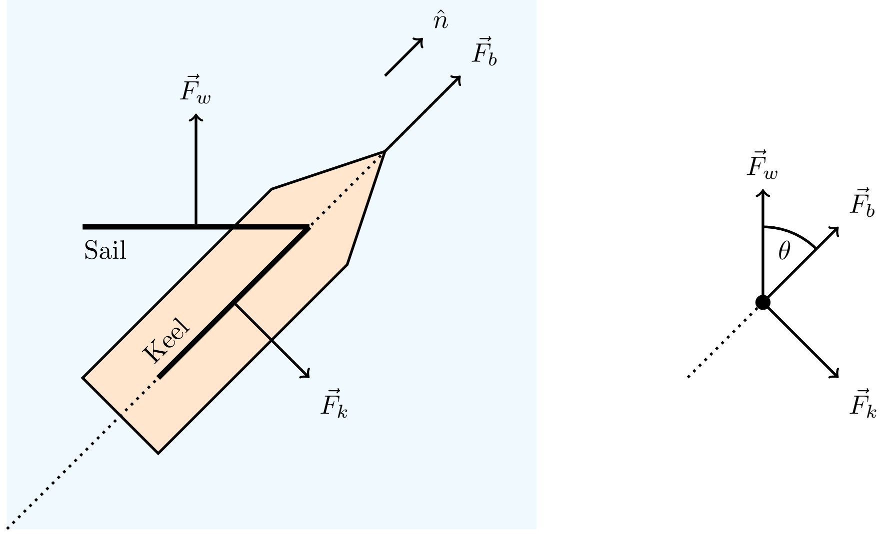

Figure 10.2: Sail boat moving in direction $\hat{n}$ with force $\vec{F}_w$ on the sail, and resistance force $\vec{F}_k$ perpendicular to the keel.

Answer to part (b)

Using Newton's second law

$$\vec{F}_w + \vec{F}_k = \vec{F}_b = m\vec{a}$$

In the direction of the boat the only force acting is

$$F_w \cos\theta = F_b = m a_b$$

i.e., we need to work out the component of $\vec{F}_w$ in the direction the boat is pointing.

$$F_w \cos\theta = \vec{F}_w \cdot \hat{n} = \vec{F}_w \cdot \frac{\vec{n}}{|\vec{n}|}$$

$$= \frac{(-50\hat{\imath} + 250\hat{\jmath}) \cdot (2\hat{\imath} + 3\hat{\jmath})}{\sqrt{2 \times 2 + 3 \times 3}} = 180 \text{ N}$$

Then we use $F_b = m a_b$ with $m = 43 + 78 = 121$ kg. Hence

$$a_b = 1.49 \text{ m s}^{-2}$$

11 Problems 3

Week 6

11.1 Vector product

Find the vector product of $\vec{a} \times \vec{b}$ where

$$\vec{a} = 2\hat{\imath} + 3\hat{\jmath}$$

$$\vec{b} = 3\hat{\imath} + 4\hat{\jmath}$$

Draw a sketch to check your answer – does the magnitude of the vector correspond to the area you expect, and is it in the direction you expect?

Answer

We need to find the determinant of the following 3×3 matrix.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

Using Equation 8.1 we get

$$\vec{a} \times \vec{b} = +(3 \times 0 - 4 \times 0)\hat{\imath} - (2 \times 0 - 3 \times 0)\hat{\jmath} + (2 \times 4 - 3 \times 3)\hat{k}$$

$$\vec{a} \times \vec{b} = -\hat{k}$$

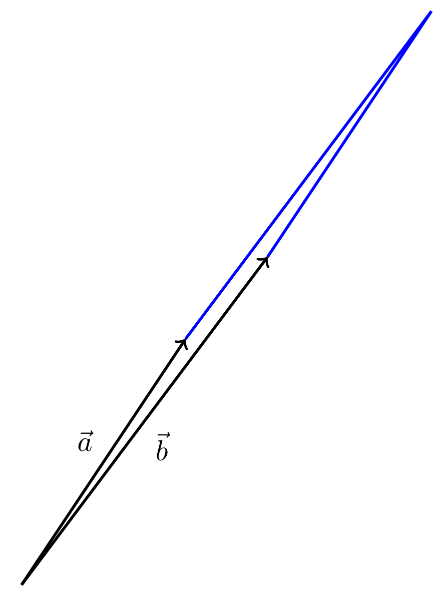

Figure 11.1: Sketch of $\vec{a}$ and $\vec{b}$. The vectors are in the $\hat{\imath}-\hat{\jmath}$ plane and given their relative positions $\vec{a}\times\vec{b}$ will be in the $-\hat{k}$ direction to give a right handed set of vectors (i.e., into the page).

11.2 Lorentz force

The force on a particle of charge q moving with velocity $\vec{v}$ due to an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is given by the *Lorentz force*, defined as follows.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

A particle of charge +1 coulomb is moving with velocity $\vec{v}$ in an electric field $\vec{E}$ and magnetic field $\vec{B}$.

a) Calculate the force on the particle if

$$\vec{v} = 8.3\hat{\imath} + 5.6\hat{\jmath} \text{ m s}^{-1}$$

$$\vec{E} = \vec{0} \text{ V m}^{-1}$$

$$\vec{B} = 1.7\hat{\imath} + 5.0\hat{\jmath} \text{ T}$$

Is any work being done on the particle by the Lorentz force in this case?

b) Calculate the force on the particle if

$$\vec{v} = -9\hat{\imath} + 6\hat{\jmath} - 3\hat{k} \text{ m s}^{-1}$$

$$\vec{E} = 6\hat{\imath} - 4\hat{\jmath} + 2\hat{k} \text{ V m}^{-1}$$

$$\vec{B} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k} \text{ T}$$

Answer to part (a)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8.3 & 5.6 & 0 \\ 1.7 & 5.0 & 0 \end{vmatrix} = (0)\hat{\imath} - (0)\hat{\jmath} + (8.3 \times 5.0 - 1.7 \times 5.6)\hat{k}$$

$$= 32\hat{k} \text{ N}$$

No work is being done because the force is perpendicular to the velocity.

Answer to part (b)

Let's first calculate $\vec{v} \times \vec{B}$ which is given by

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -9 & 6 & -3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= (6-6)\hat{\imath} - (-9+9)\hat{\jmath} + (18-18)\hat{k}$$
$$= \vec{0}$$

because $\vec{v}$ is parallel to $\vec{B}$. Therefore,

$$\vec{F} = q(\vec{E} + \vec{0}) = 6\hat{\imath} - 4\hat{\jmath} + 2\hat{k} \text{ N}$$

All three vectors parallel, solution parallel to $\vec{E}$.

11.3 Circular motion in a synchrotron – angular SUVAT

Synchrotrons are used to generate high intensity X-ray beams. The Diamond light source at Harwell in Oxfordshire is one of the most powerful X-ray sources available for physics research. At Diamond, electrons are first accelerated in a linear accelerator and then injected into a synchrotron ring where they are further accelerated (by a constant angular acceleration) up to their final speed, which is a substantial fraction of the speed of light.

Suppose the electrons are travelling with a speed v when they first enter the synchrotron ring which has a radius R. This means they have an initial angular speed $\omega_0 = v/R$ (measured in rad s^{-1}) and are subjected to a constant angular acceleration α measured in rad s^{-2}, until they reach their final speed. You may also assume that the initial angle that the electrons enter the ring $\phi_0 = 0$ rad. You should ignore relativistic effects.

(a) Write down an equation describing the angular velocity ω of an electron as a function of time since its injection.
(b) By integration, find an equation describing the angular position ϕ (in radians) of the electron as a function of time.
(c) Using the result of the previous part, find the total distance (in m) that the electron will move around the synchrotron ring as a function of time.

Answer to part (a)

We start by constructing an algebraic expression describing the angular velocity increasing linearly as a function of time, i.e., with a constant angular acceleration α

$$\omega(t) = \omega_0 + \alpha t$$

Answer to part (b)

By writing the angular velocity as $d\phi/dt$, we can find an expression for the angular position ϕ as a function of time.

$$\phi = \int_{t'=0}^{t'=t} \omega(t')dt'$$

$$= \int_{t'=0}^{t'=t} (\omega_0 + \alpha t')dt'$$

$$= \omega_0 t + \frac{1}{2}\alpha t^2$$

Answer to part (c)

The distance travelled $s(t)$ around the ring is the angular coordinate multiplied by the radius R of the ring. So

$$s(t) = R\phi(t) = R\left(\omega_0 t + \frac{1}{2}\alpha t^2\right)$$

11.4 Moment of inertia of a cylinder

Note: I have changed the wording of this question a little bit after feedback from the problems class in an attempt to make it clearer. Please get in touch if anything isn't clear; I am happy to discuss it with you.

Consider a thin cylindrical shell of density ρ, length L, radius a, and thickness δa, where δa is small.

(a) Show that the mass of the shell is given by

$$\delta m = \rho L 2\pi a \delta a + \underbrace{\rho L \pi (\delta a)^2}_{\text{we can ignore this term (see below)}}$$

and its moment of inertia about the axis of the cylinder is given by

$$\delta I = \rho L 2\pi a^3 \delta a + \underbrace{\rho L \pi a^2 (\delta a)^2}_{\text{we can ignore this term (see below)}}$$

Hence, in the limit as $\delta a \to 0$,

$$\frac{dI}{da} = \rho L 2\pi a^3$$

(b) The moment of inertia about the axis of a uniform, thick cylindrical tube of density ρ, length L, inner radius r_i and outer radius r_o can be calculated by considering the tube as the sum of many thin cylindrical shells. Explain, with the aid of a diagram, why the moment of inertia of the tube is given by

$$I = \int_{I_i}^{I_o} dI = \int_{a=r_i}^{a=r_o} \frac{dI}{da} da$$

and hence show that

$$I = \frac{\pi \rho L}{2}(r_o^4 - r_i^4)$$

(c) Find an expression for ρ in terms of M, r_i, r_o, and L, and show that

$$I = \frac{1}{2} M (r_i^2 + r_o^2)$$

(d) Write down expressions for the linear and rotational kinetic energy of such a cylinder when the centre of mass has a speed of v_{com}. Assume the cylinder is rolling and not slipping.

Hint: you can write $v_{\text{com}} = \omega r_o$.

(e) Such a hollow cylinder is released at the top of a slope and rolls down the slope, under gravity, without slipping. Show that the speed of the centre of mass after the cylinder has dropped by a vertical distance h is given by

$$v_{\text{com}} = \sqrt{\frac{4gh}{3 + \left(\frac{r_i}{r_o}\right)^2}}$$

(f) A hollow cylinder of mass 1 kg with inner radius 5 cm and outer radius 8 cm is rolling along the ground; the centre of mass has a speed of 1.6 m s^{-1}.

i. What is the angular velocity of the cylinder?
ii. What is the linear kinetic energy of the cylinder?
iii. What is the rotational kinetic energy of the cylinder?

Answer to (a)

The mass of the thin shell, shown in Figure 11.2, is given by the product of its circumference, thickness, length, and density

$$\begin{aligned}\delta m &= (\pi(a+\delta a)^2 - \pi a^2) \times (L) \times (\rho) \\ &= \rho L 2\pi a \delta a + \rho L \pi (\delta a)^2\end{aligned}$$

which is the answer required.

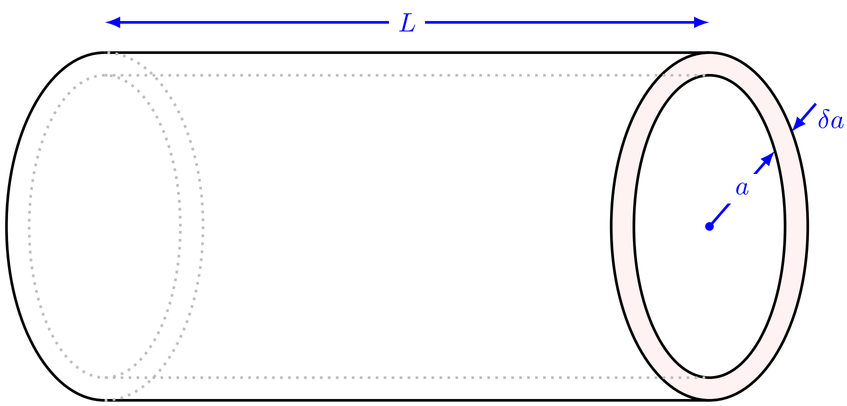

Figure 11.2: Thin cylindrical shell of inner radius a and thickness δa.

In lecture we noted that the moment of inertia of a hoop is given by

$$\delta I = a^2 \delta m$$

because all of the mass is a distance a from the rotation axis. Hence

$$\delta I = 2\pi a^3 \rho L \delta a + \rho L \pi a^2 (\delta a)^2$$

Rearranging this gives

$$\frac{\delta I}{\delta a} = 2\pi a^3 \rho L + \rho L \pi a^2 \delta a$$

Taking limit at $\delta a \to 0$ gives

$$\frac{dI}{da} = 2\pi a^3 \rho L$$

Note: in future problems and lecture courses you will typically just write down the answer ignoring the terms of order $(\delta a)^2$, knowing that these will disappear in the limit as $\delta a \to 0$.

Answer to (b)

We can build up a thick cylinder out of individual, thin cylindrical shells, as shown in Figure 11.3.

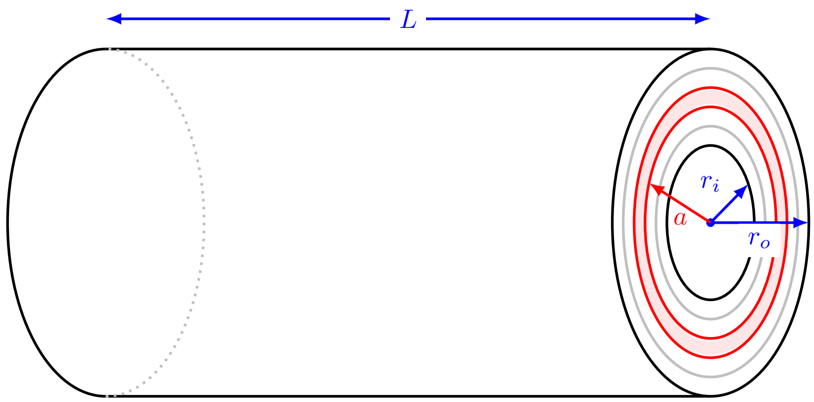

Figure 11.3: Cylindrical shells making up a thick cylinder of inner radius r_i and outer radius r_o. Each individual shell (shaded in red) has a moment of inertia given in part (a).

We need to add up all of the moments of inertia of the individual shells to get the total moment of inertia of the thick cylinder. This is given by

$$I = \int_{I_i}^{I_o} dI = \int_{a=r_i}^{a=r_o} \frac{dI}{da} da$$

$$I = 2\pi \rho L \int_{a=r_i}^{a=r_o} a^3 da$$

$$I = 2\pi \rho L \left[\frac{1}{4}a^4\right]_{r_i}^{r_o}$$

$$I = \frac{\pi \rho L}{2}(r_o^4 - r_i^4)$$

Answer to (c)

The total mass of the cylinder is equal to its density times its volume.

$$M = \rho L \pi (r_o^2 - r_i^2)$$

which we can rearrange for the density, ρ, as follows

$$\rho = \frac{M}{L\pi(r_o^2 - r_i^2)}$$

Hence

$$I = \frac{\pi L}{2} \frac{M}{L\pi(r_o^2 - r_i^2)}(r_o^4 - r_i^4)$$

$$I = \frac{1}{2}M\frac{(r_o^2 - r_i^2)(r_o^2 + r_i^2)}{(r_o^2 - r_i^2)}$$

$$I = \frac{1}{2}M(r_i^2 + r_o^2)$$

Answer to (d)

The diagram is similar to the example we went through in the last lecture, and is shown in Figure 8.15.
Linear kinetic energy is given by

$$\text{Linear KE} = \frac{1}{2}Mv_{\text{com}}^2$$

Rotational kinetic energy is given by

$$\text{Rotational KE} = \frac{1}{2}I\omega^2$$

where ω is the angular velocity of the cylinder. Since the cylinder is not slipping $\omega = v_{\text{com}}/r_o$.

$$\text{Rotational KE} = \frac{1}{2}I\left(\frac{v_{\text{com}}}{r_o}\right)^2 = \frac{1}{4}Mv_{\text{com}}^2\left(1 + \frac{r_i^2}{r_o^2}\right)$$

Answer to (e)

As the cylinder rolls down the slope under gravity, the change in its gravitational PE is equal to the increase in KE. From the previous part, the total KE is

$$\text{Total KE} = \frac{1}{4} M v_{\text{com}} \left(3 + \frac{r_i^2}{r_o^2}\right)$$

while the increase in gravitational PE is Mgh. Setting these two energy changes equal to each other gives,

$$\frac{1}{4} v_{\text{com}} \left(3 + \frac{r_i^2}{r_o^2}\right) = gh$$

$$v_{\text{com}} = \sqrt{\frac{4gh}{3 + \left(\frac{r_i}{r_o}\right)^2}}$$

as required.

Answer to (f)

i.

$$\omega = \frac{v_{\text{com}}}{r_o} = 20 \text{ rad s}^{-1}$$

ii.

$$\text{Linear KE} = 1.28 \text{ J}$$

iii.

$$\text{Rotational KE} = 0.64 \left(1 + \frac{25}{64}\right) = 0.89 \text{ J}$$

Part III

Workshops

12 Workshop guide

12.1 Purpose of workshops

Workshops are a place to get to know your colleagues on the course and offer an opportunity to **discuss physics** as a part of solving problems.

You will practice problem solving in groups. In particular, you will learn how to analyse a problem, draw diagrams, write down appropriate equations, and clearly present your workings. You should bear in mind the following points:

- Solutions should be more than just strings of equations; there should be a commentary to *explain* the solution.
- A diagram can help to answer questions; make sure your diagram is clear and large enough to label and write on. Diagrams can be too small – they are rarely too big!

12.2 Workshop attendance

Attendance at workshops is required to receive a mark for your submitted workshop assignment (part 2 question). Absence from the workshop will result in zero marks unless you have reported your absence with a good reason. More information about absences can be found on the School of Physics Blackboard organisation which includes a link to the absence form.

12.3 Structure of workshops

The workshops are two hours long: check your individual timetable for time and location.

The workshops questions are divided into two parts as follows.

- Part 1: Preliminary questions.
- Part 2: Problem-solving question.

12.3.1 Part 1: Preliminary questions

These are “warm up”, practice questions which are relatively short.

These questions will be posted before the workshops to give you the opportunity to have a brief look through them beforehand if you wish: there is no obligation to do so. We have made this decision to post these questions beforehand by consideration of EDI issues and previous student feedback.

The preliminary questions do not count towards your final grade. During the workshop, your workshop leader will ask you to discuss with your colleagues *what a student might find difficult* for a given problem in Part One. For example, you can discuss:

- What concepts are needed to solve the problem?
- Was a diagram helpful?
- What equations might help?
- Are there any unit conversions?
- Is there any information included that is not needed to solve the question?
- Is there any assumption that you need to make to solve the problem?

These questions are helpful to consider before solving any problem.

12.3.2 Part 2: Problem-solving question

The Part 2 question will be released during your workshop by your workshop leader.

This question is more involved and open in structure.

This question is submitted to Blackboard for assessment either as an individual submission or a a group submission (i.e. single submission on behalf of the group). Refer to the “Workshop Schedule”. Note that Workshop 0 (during week 1) is formative.

You might be able to do the problem solving questions using pre-University knowledge. However, it will require some novel approaches and discussions with your colleagues. Usually, the main challenge is relating the problem description to the physical model needed to solve the questions.

- The first step to solving a complex problem is to draw a diagram and work out what you already know (rather than worrying about what you do not know!). Write down formulas and physics related to the problem and start using them to analyse and solve it.
- Hints will be available to help you on your way, but you should first discuss what you have already determined before asking for a hint.

We hope you enjoy the workshops!

12.4 Submitting your work

12.4.1 Preparation of part 2 solution

The full solutions to 'Part Two' should be submitted via Blackboard as a single PDF file after the session following the guidance below:

- Read carefully the instructions for converting multiple pages of handwritten notes into a single PDF file.
- Write your solutions clearly as possible.
- Include your name and page number on every page.

- All your hand written work should be scanned or photographed and saved as PDF files.
- Combine the files in the correct order and save as a single PDF with the appropriate name (for example "Workshop0-YOUR_NAME").
- For group submissions make sure to include the names of all members of the group who made the contributions.

12.4.2 Submitting your part 2 solution

In the "Unit Assessment" folder on Blackboard there are folders for each workshop week that have links to the Submission Point for your workshop group. *Note*: the submission links will depend on the day of your workshop.

Please familiarise yourself with the Blackboard assessment guide which includes information on how to submit assessments.

The deadline for submission is 5 pm a day after your workshop session

12.5 Marking scheme and feedback

12.5.1 Breakdown of marks

The following information expands on the grading criteria you will see on Blackboard.

12.5.1.1 Demonstration of understanding (5 points, 25%)

- **0** – No solutions provided
- **E** – Little or no narrative provided. Some perfunctory mathematical “therefore” symbols used, but reader left making leaps of logic.
- **D** – An attempt made to explain some points in the solution, but otherwise absent. Little more than short perfunctory statements offered to explain logic.
- **C** – Some steps have explanatory notes, with links between equations as needed. Some steps missing explanation or leaps in logic made.
- **B** – Most steps in the solution have an explanatory note with links between equations as needed. Not too detailed, but enough to guide the reader.
- **A** – A clear progression shown from step to step, demonstrating understanding, links between equations, derivations (where appropriate) explained. An essay is not needed! Just logical steps.

12.5.1.2 Organisation of solutions (5 points, 25%)

- **0** – Nothing presented/completely unreadable.
- **E** – Unacceptable organisation: Very little clarity in the solution presented, few logical steps between the solution, little or no working shown. Results difficult to identify in the solution.
- **D** – Poor organisation: Large aspects of solution unclear, working not shown, “final answer” not clear from the rest of the work. Poorly presented diagrams, unclear/missing labels.
- **C** – Fair organisation: Some aspects of the solution unclear, or “working” is not distinct from “final answer”, or some working not shown. Where diagrams used, most labels apparent and presented with reasonable clarity.
- **B** – Good organisation: Solution laid out clearly, some elements unclear, but largely a clear solution, with most ‘working’ shown. Solutions presented using the space on the page with well labelled diagrams where appropriate.
- **A** – Excellent organisation, clear delineation between presented solutions, explanations and rough working, with all results (including intermediate results) clearly displayed. All working shown, any mistakes/errors crossed through without total erasure. Solutions presented using the space on the page with well labelled diagrams where appropriate.

12.5.1.3 Application of knowledge - solution (10 points, 50%)

- **0** – No solutions provided
- **E** – A poor solution or unfinished solution with very little correct, however an attempt was made. >20% in worked solution.

- **D** – Substantial errors, or an incomplete solution with correct elements; or "correct answers" given but no route presented in the working. >40% in worked solution
- **C** – Solution mostly correct, a few errors carried forward, one unit error. >60% in worked solution.
- **B** – A largely correct solution, but precision not addressed >80% in worked solution.
- **A** – A flawless solution with due consideration to units and precision. "Full marks" in worked solution.

12.5.2 Using your rubric score

It is very common for students to look for "why I lost marks". Let us first dispel that idea; the rubric is constructed in such a way that **marks are only ever awarded and never taken away**. When you look at the rubric below you will see that each stage builds on the last. Therefore, when you see your score, you should reflect on which aspects of that stage your work best fulfils and identify which aspects you need to work on in order to advance to the next stage.

Please be reassured that **students never lose marks** in the course of grading an assessment. You are awarded marks for demonstrating understanding, but you are never penalised for making a mistake. This is why it is important to always show your working.

Also note that, in demonstrating understanding, credit is never fully awarded for the final answer on its own - it must be supported by appropriate working and reasoning. Above all, your mark is only a useful form of feedback if you properly reflect on it and understand how it was arrived at.

12.5.3 Accessing your workshop marks and feedback

It is best to view your feedback from a desktop computer or a laptop. Smartphones and tablets tend not to display feedback well.

Click on the "My Grades and Feedback" link to access your workshop marks and feedback.

Here we aim to give you more insight into how you're scoring against the rubric: this should be used as a form of feedback to help you improve for the future.

Detailed instructions on how to access your feedback can be found here: Accessing marks and feedback in Blackboard or Turnitin.

12.5.4 Workshop answers

Model workshop solutions to part 1 for each workshop will be posted in the following week.

13 Workshop 0

Week 1

This workshop is a reflection on A-level mechanics problems.

The purpose of the workshop is to get used to the workshop environment, get to know your course colleagues, **discuss physics**, and begin solving physics problems.

The problems are based on A-level mechanics. All problems are solvable using that knowledge. Discussions with colleagues and referring to notes, books and other resources are encouraged. You should bear in mind the following points:

- Solutions should be more than just strings of equations; there should be explanation and commentary to explain the solution.
- A diagram can help to answer questions; make sure your diagram is clear and large enough to label and write on. Diagrams can be too small – they are rarely too big!

13.1 Part 1 – Preliminary questions

Formative (i.e., not assessed), for practice, and answers do not need to be submitted

These are "primer" questions. The problems are based on A-level mechanics, and the solutions can be found using your mechanics notes. We will also cover this content later in the lectures, so don't worry if you can't remember everything now.

13.1.1 Force diagram

Draw force diagrams for a car in the following scenarios.

a) Going uphill at constant velocity.
b) Going downhill at constant velocity.
c) Stationary on a hill.

Answer

This is an application of Newton's first law. Since the car is either "at rest" or "moving at constant speed", all forces are in balance. We also only consider "external forces"; i.e., we ignore "friction in wheel bearings"! For simplicity's sake, we combine all "drag" forces.

a)

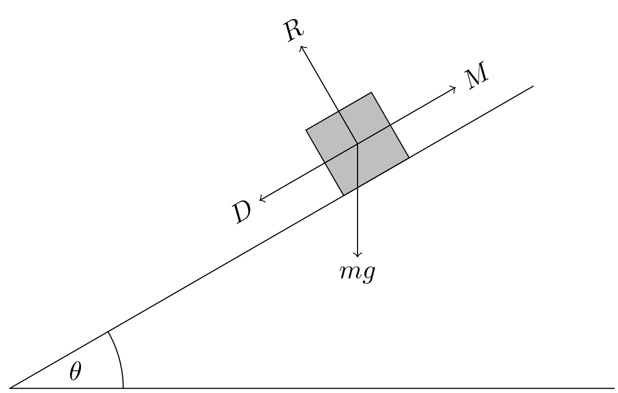

Figure 13.1: Force diagram for part (a), constant speed uphill.

- Constant speed uphill.
- $R = mg \cos \theta$ is the normal reaction force.
- M is the motive force.

- D is the drag force due to friction and air resistance.
- The motive force and drag force are balanced, $M = D + mg\sin\theta$.

b)

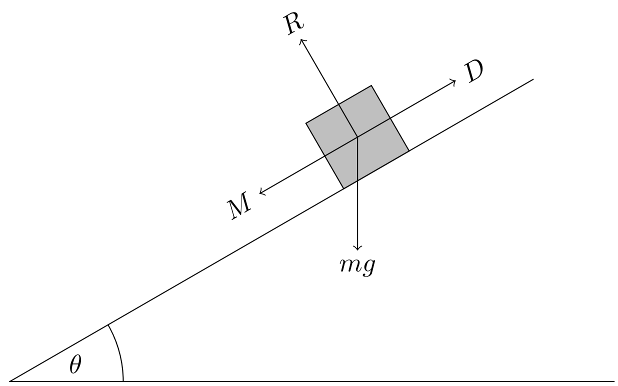

Figure 13.2: Force diagram for part (b), constant speed downhill.

- Constant speed downhill.
- $R = mg\cos\theta$ is the normal reaction force.
- M is the motive force (if needed).
- D is the drag force due to friction, air resistance, and braking (if needed).
- The motive force and drag force are balanced, $D = mg\sin\theta + M$.

c)

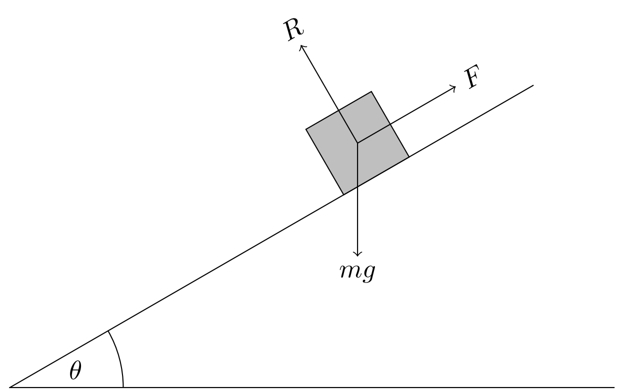

Figure 13.3: Force diagram for part (c), stationary car.

- Stationary car.

- $R = mg\cos\theta$ is the normal reaction force.
- F is the friction force.
- The friction force and the component of gravitational force along the plane are balanced, $F = mg\sin\theta$.

Note: These answers do contain some information that wasn't asked for in the question, to help your understanding. We would only expect you to provide the information that is asked for in an exam!

13.1.2 Drone

A drone of mass 1.0 kg rises with constant acceleration from rest to a height of 50 m in 10 s. Calculate the thrust (force) exerted by the rotors of the drone.

Answer

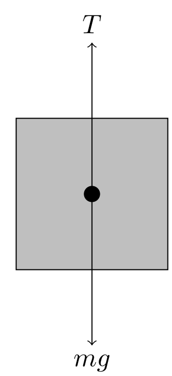

Figure 13.4: Drone with gravitational force acting downwards and thrust force acting upwards.

This is an application of Newton's equations of motion to find the acceleration. Using

$$s = ut + \frac{1}{2}at^2$$

where the initial velocity $u = 0$ m s^{-1} and rearranging to get

$$a = \frac{2s}{t^2}.$$

We can then apply Newton's second law, $F = ma$. Looking at Figure 13.4, the net upward force providing the acceleration is $F = T - mg$.

$$T - mg = ma$$

$$T = mg + ma$$

$$T = m\left(\frac{2s}{t^2} + g\right)$$

Substituting in the numbers gives

$$T = 1 \times \left(\frac{2 \times 50}{10^2} + 9.8\right) = 11 \text{ N}.$$

13.1.3 Bicycle torque

A bicycle hub cassette must be tightened to a torque of 50 Nm. If you have a spanner of length 25 cm, what is the perpendicular force required to obtain the correct torque?

Answer

To answer this question you need to recognise what a "moment" is. This is also an opportunity to mention dimensional analysis.

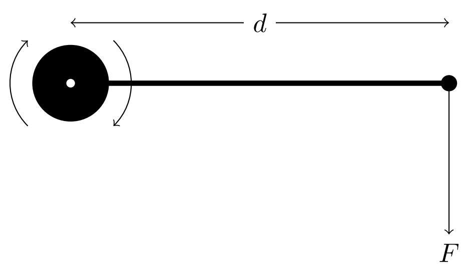

Figure 13.5: A spanner of length d and a force F acting perpendicular to the spanner.

Torque, as illustrated in Figure 13.5, is $\tau = Fd$.

$$F = \frac{\tau}{d} = \frac{50}{0.25} = 200 \text{ N}.$$

13.2 Part 2 – Problem-solving question

The problem solving question will be provided during the workshop session.

Full solution should be submitted as a single PDF as a group submission under "Unit Assessment" link on Blackboard. One upload per group.

14 Workshop 1

Week 3

14.1 Part 1 - Preliminary questions

Formative (i.e., not assessed), for practice, and does not need to be submitted

These first questions are designed as revision of some of the maths you need during the mechanics course: calculus and vectors. It is always helpful to practise these techniques as they often appear in other parts of Physics!

14.1.1 Velocity to position

The velocity of a particle is given by:

$$\vec{v} = (5t\hat{\imath} + 6t^2\hat{\jmath} + 3\hat{k}) \text{ m s}^{-1}$$

where the numeric constants carry appropriate dimensions. Find an expression for the position vector of the particle $\vec{r}(t)$, given that $\vec{r}(0) = (3\hat{\imath} + 2\hat{\jmath} + \hat{k})$ m.

Answer

We need to integrate

$$\vec{v} = \frac{d\vec{r}}{dt}$$

to get $\vec{r}(t)$.

$$\vec{r} = \int \left(5t\hat{\imath} + 6t^2\hat{\jmath} + 3\hat{k}\right) dt$$

$$\vec{r} = \left(\frac{5}{2}t^2\hat{\imath} + 2t^3\hat{\jmath} + 3t\hat{k}\right) + \vec{c}$$

where $\vec{c}$ is a constant vector. We can find $\vec{c}$ by substituting in the initial condition for $\vec{r}(0)$ at $t = 0$. This gives $\vec{c} = \vec{r}(0)$.

$$\vec{r}(t) = \left[\frac{1}{2}(5t^2 + 6)\hat{\imath} + 2(t^3 + 1)\hat{\jmath} + (3t + 1)\hat{k}\right]$$

14.1.2 Work and angle

A force $\vec{f} = (5\hat{\imath} + 7\hat{\jmath} + 9\hat{k})$ N displaces a second particle from the origin to position vector $\vec{r} = (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ m. Using the scalar product, calculate the work done on the particle by the force, $W = \vec{f} \cdot \vec{r}$ and find the angle between $\vec{f}$ and $\vec{r}$.

Answer

$$W = \vec{f} \cdot \vec{r} = 5 \times 2 + 7 \times 2 + 9 \times 2 = 42 \text{ J}$$

$$\vec{f} \cdot \vec{r} = |\vec{f}||\vec{r}| \cos\theta$$

$$|\vec{f}| = \sqrt{5^2 + 7^2 + 9^2} = 12.45 \text{ N}$$

$$|\vec{r}| = \sqrt{2^2 + 2^2 + 2^2} = 3.46 \text{ m}$$

$$\cos\theta = \frac{\vec{f} \cdot \vec{r}}{|\vec{f}||\vec{r}|} = \frac{42}{12.45 \times 3.46} = 0.9750$$

$$\theta = 12.8^\circ$$

14.1.3 Constant force

The next few questions are conceptual, and aimed at testing your understanding of Newton's laws. They are also multiple choice. We haven't covered the third law yet, but have a go anyway, as you should be familiar with the ideas from school.

A student exerts a constant horizontal force on a large box, causing it to move across a horizontal floor at constant speed v_0. The constant force applied by the student:

A) Is greater than either the weight of the box or the total force which resists its motion.
B) Has the same magnitude as the weight of the box.
C) Is greater than the total force which resists the motion of the box.
D) Has the same magnitude as the total force which resists the motion of the box.
E) Is greater than the weight of the box.

Answer

D - no acceleration, so forces must be balanced.

14.1.4 Constant force $\times 2$

The same student doubles the constant horizontal force exerted. The box then moves:

A) For a while at a constant speed, greater than the speed v_0, then with a speed that increases thereafter.
B) For a while with an increasing speed, then with a constant speed thereafter.
C) With a continuously increasing speed.
D) With a constant speed that is double the speed v_0, in the previous question.
E) With a constant speed that is greater than the speed v_0, but not necessarily twice as great.

> Answer
>
> C - forces unbalanced, so box accelerates.

14.1.5 Truck towing

A large truck breaks down, and receives a push back into town by a small compact car.

a) While the car, pushing the truck, is speeding up to a cruising speed:

   A) The amount of force with which the car pushes on the truck is equal to that with which the truck pushes back on the car.
   B) The amount of force with which the car pushes on the truck is smaller than that with which the truck pushes back on the car.
   C) The amount of force with which the car pushes on the truck is greater than that with which the truck pushes back on the car.
   D) The car's engine is running so that car pushes against the truck, but the truck's engine is not running so that truck cannot push back against the car. The truck is pushed forward simply because it is in the way of the car.
   E) Neither the car nor the truck exerts any force on the other. The truck is pushed forward simply because it is in the way of the car.

> Answer
>
> A - by Newton's third law, the car and truck push on each other with equal and opposite forces.

b) After the car reaches the constant cruising speed at which its driver wishes to push the truck:

A) The amount of force with which the car pushes on the truck is smaller than that with which the truck pushes back on the car.
B) The amount of force with which the car pushes on the truck is greater than that with which the truck pushes back on the car.
C) The car's engine is running so that car pushes against the truck, but the truck's engine is not running so that truck cannot push back against the car. The truck is pushed forward simply because it is in the way of the car.
D) The amount of force with which the car pushes on the truck is equal to that with which the truck pushes back on the car.
E) Neither the car nor the truck exerts any force on the other. The truck is pushed forward simply because it is in the way of the car.

Answer

D - as above, although the forces are now lower, as we only need to overcome friction. Without friction there would be no forces acting, and we would only be considering Newton's first law.

14.1.6 River swim

(a) A swimmer crosses a river by swimming perpendicular to the bank. The river is flowing uniformly at 1 m s^{-1}. The swimmer swims at a steady 1 m s^{-1} relative to the water. What is the swimmer's velocity relative to the frame of the bank, and whereabouts do they land on the other bank, if the river is 100 m wide? How long does the crossing take?

Answer to (a)

A sketch of the problem is shown in Figure 14.1.

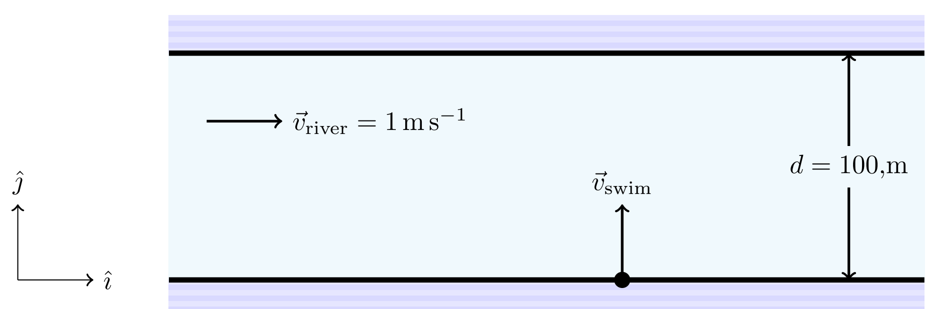

Figure 14.1: Swimming across a river.

The velocity relative to the river bank is given by

$$\vec{v}_{\text{bank}} = \vec{v}_{\text{swim}} + \vec{v}_{\text{river}}$$

as shown in Figure 14.2.

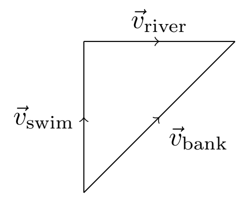

Figure 14.2: Velocity relative to the river bank.

$$\vec{v}_{\text{bank}} = |\vec{v}_{\text{swim}}|\,\hat{\imath} + |\vec{v}_{\text{river}}|\,\hat{\jmath} = (\hat{\imath} + \hat{\jmath}) \text{ m s}^{-1}$$

For the crossing time, consider the perpendicular component

$$d = |\vec{v}_{\text{swim}}| \times t$$

$$\implies t = \frac{100}{1} = 100 \text{ s}$$

Position vector of landing relative to starting point on the bank $\vec{r}_{\text{start}} = (0, 0)$ is given by

$$\vec{r}_{\text{end}} = \vec{v}_{\text{bank}} \times t = 100(\hat{\imath} + \hat{\jmath}) \text{ m}$$

You could also obtain this geometrically because the trajectory is 45° to the bank.

(b) The same swimmer attempts to cross the river again but, on this occasion, they aim to land directly opposite their starting position. In what direction should they swim, and how long will the crossing take, if they swim at 2 m s^{-1} relative to the water?

Answer to (b)

This is similar to before (see Figure 14.3),

$$\vec{v}_{\text{bank}} = \vec{v}_{\text{swim}} + \vec{v}_{\text{river}},$$

but now we want $\vec{r}_{\text{bank}}$ parallel to $\vec{\jmath}$ and $|\vec{v}_{\text{swim}}| = 2$ m s^{-1}.

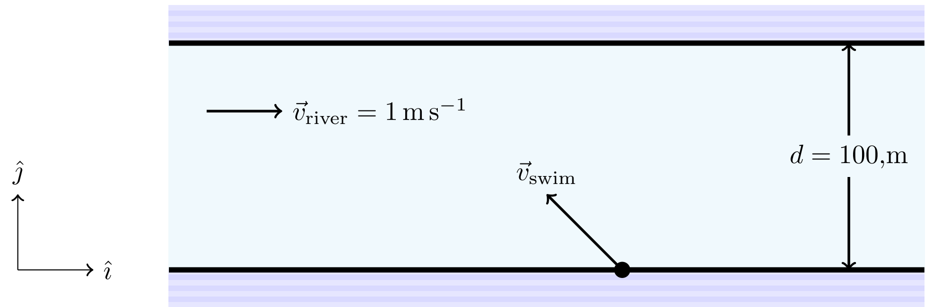

Figure 14.3: Swimming across a river so that the swimmer lands directly opposite their starting point.

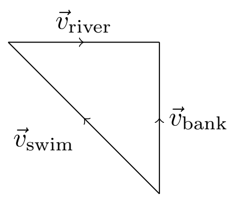

Figure 14.4: Velocity relative to the river bank.

From Figure 14.4, we can see that

$$|\vec{v}_{\text{swim}}|^2 = |\vec{v}_{\text{river}}|^2 + |\vec{v}_{\text{bank}}|^2$$

$$4 = 1 + |\vec{v}_{\text{bank}}|^2$$

Substituting in the numbers gives

$$\vec{v}_{\text{bank}} = \sqrt{3}\hat{\jmath} \text{ m s}^{-1}$$

and a crossing time of

$$t = \frac{100}{\sqrt{3}} = 57.7 \text{ s}$$

The direction is given by θ,

$$\cos\theta = \frac{|\vec{v}_{\text{bank}}|}{|\vec{v}_{\text{swim}}||} = \frac{\sqrt{3}}{2} = 0.866$$

$$\theta = 30^\circ$$

14.2 Part 2 – Problem-solving question

Summative (i.e., counts 3.75% towards final grade), please submit PDF as a group

The problem solving question will be provided during the workshop session.

15 Workshop 2

Week 5

15.1 Part 1 - Preliminary questions

Formative (i.e., not assessed), for practice, and does not need to be submitted

15.1.1 Work done in circular motion

What is the rate at which work is done on a particle executing motion in a circle at a constant speed?

> Answer
>
> Resultant work done during circular motion at uniform speed is zero, since the force is acting perpendicular to the direction of motion and so $\vec{F} \cdot \vec{dr} = 0$.

15.1.2 Neutron colliding with carbon nucleus

A neutron of mass m_n and speed v_{ni} undergoes a head-on, perfectly elastic collision with a carbon nucleus of mass m_C initially at rest. You may assume Newtonian mechanics applies.

a) What are the final velocities of the neutron and carbon nucleus?
b) What fraction of its initial energy does the neutron lose?

> Answer to (a)
>
> The question specifies a *head-on* collision. This implies that the recoil velocities, after the collision, will be along the same line as the initial velocity. So we are dealing with a one dimensional elastic collision here. Conservation of momentum and conservation of energy

can both be applied. (Conservation of energy because the collision is *elastic*).
A carbon nucleus is (to a good approximation, assuming the most common isotope of carbon) 12 times more massive that a neutron so we expect the neutron to "bounce back" with close to its initial speed while the carbon nucleus moves away rather more slowly in the original direction.
So, taking v_C in the original direction and v_{nf} in the opposite direction, we have from conservation of momentum,

$$m_n v_{ni} = m_C v_{Cf} - m_n v_{nf}. \tag{15.1}$$

Conservation of energy gives

$$\frac{1}{2} m_n v_{ni}^2 = \frac{1}{2} m_C v_{Cf}^2 + \frac{1}{2} m_n v_{nf}^2. \tag{15.2}$$

We solve these two equations for v_{Cf} and v_{nf}. There are different ways you can do this, including the following. Rearranging and squaring Equation 15.1 gives

$$m_C^2 v_{Cf}^2 = (m_n v_{ni} + m_n v_{nf})^2.$$

We can get another expression for $m_C^2 v_{Cf}^2$ from Equation 15.2,

$$m_C^2 v_{Cf}^2 = m_C (m_n v_{ni}^2 - m_n v_{nf}^2).$$

Equating these, we get

$$m_n^2 (v_{ni} + v_{nf})^2 = m_C m_n (v_{ni}^2 - v_{nf}^2)$$
$$m_n (v_{ni} + v_{nf}) = m_C (v_{ni} - v_{nf})$$
$$v_{ni} (m_C - m_n) = v_{nf} (m_n + m_C)$$
$$v_{nf} = \frac{m_C - m_n}{m_C + m_n} v_{ni}$$
$$v_{nf} = \frac{11}{13} v_{ni}$$

And then, from Equation 15.1,

$$v_{Cf} = \frac{m_n v_{ni}}{m_C} \left(1 + \frac{11}{13}\right) = \frac{24 m_n v_{ni}}{13 m_C}$$
$$v_{Cf} = \frac{2}{13} v_{ni}$$

assuming the ratio of the particle masses is exactly 12. These values confirm our initial expectations since v_{nf} is close to v_{ni} and the carbon nucleus speed is considerably smaller.

Answer to (b)

The fraction of kinetic energy lost by the neutron is

$$\frac{\frac{1}{2}m_n v_{ni}^2 - \frac{1}{2}m_n v_{nf}^2}{\frac{1}{2}m_n v_{ni}^2}$$

$$= \frac{1 - \left(\frac{11}{13}\right)^2}{1} = \frac{48}{169} = 28\%$$

15.1.3 Vector products

Find the vector product $\vec{a} \times \vec{b}$ of the following vectors.

(a) $\vec{a} = 4\hat{\imath}$ and $\vec{b} = 6\hat{\imath} + 6\hat{\jmath}$.
(b) $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$.

Note on vector product

If you haven't seen the vector product before, we will go through this in lectures. You can find out more by looking at Section 8.1 of the notes. Here is the information you need to answer this question.
The vector product of two vectors $\vec{a}$ and $\vec{b}$ produces a third vector $\vec{c}$ that is perpendicular to both $\vec{a}$ and $\vec{b}$. The magnitude of $\vec{c}$ is equal to the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.
There are various ways to calculate $\vec{c}$ including the *determinant method* described below.

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

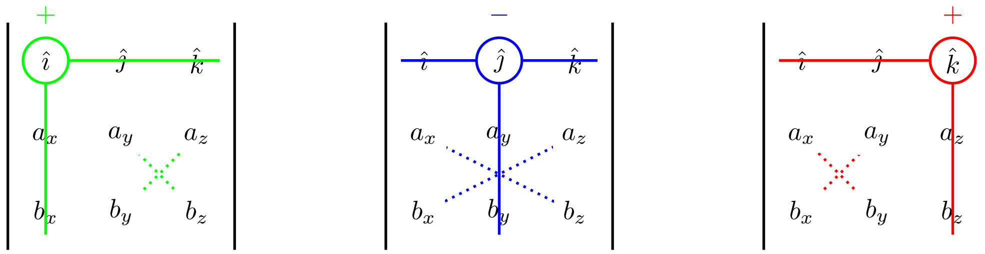

Figure 15.1: Illustration of the pattern you can remember for calculating the determinant. Each circled unit basis vector, with sign above it, is multiplied by the *cofactor* indicated by the dotted lines to get the vector product.

$$\vec{a} \times \vec{b} = +(a_y b_z - b_y a_z)\hat{\imath} - (a_x b_z - b_x a_z)\hat{\jmath} + (a_x b_y - b_x a_y)\hat{k}$$

Note the "−" sign associated with the $\hat{\jmath}$ term.

Answer to (a)

You can either remember the pattern we look at in lectures, or do the cross product term-by-term. We'll write it down term-by-term here

$$4\hat{\imath} \times (6\hat{\imath} + 6\hat{\jmath}) = 4\hat{\imath} \times 6\hat{\imath} + 4\hat{\imath} \times 6\hat{\jmath}$$

because the vector product is *distributive.*

$$\hat{\imath} \times \hat{\imath} = \vec{0}$$
$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

Which gives the final answer

$$\vec{a} \times \vec{b} = 24\hat{k}$$

Answer to (b)

This is probably easiest to do using the determinant method (if you're familiar with that) or the pattern / formula we looked at in lectures. However, to be clear, we'll write it down term-by-term as this makes it clear what we are doing. Remember that the cross product is distributive.

$$(2\hat{\imath} + 3\hat{\jmath} - \hat{k}) \times (3\hat{\imath} + \hat{\jmath} + 4\hat{k})$$
$$= 2\hat{\imath} \times \hat{\jmath} + 2\hat{\imath} \times 4\hat{k} + 3\hat{\jmath} \times 3\hat{\imath} + 3\hat{\jmath} \times 4\hat{k} - \hat{k} \times 3\hat{\imath} - \hat{k} \times \hat{\jmath}$$

where we have removed the $\hat{i} \times \hat{i}$, $\hat{j} \times \hat{j}$ and $\hat{k} \times \hat{k}$ terms because they are zero. We can then look at the cross product of the basis vectors individually remembering that the cross product produces a right-handed set of vectors.

$$\vec{a} \times \vec{b} = 2\hat{k} - 8\hat{\jmath} - 9\hat{k} + 12\hat{\imath} - 3\hat{\jmath} + \hat{\imath}$$

$$= 13\hat{\imath} - 11\hat{\jmath} - 7\hat{k}$$

15.1.4 Person pulling sled

A girl pulls a sled in which her brother sits, over the snow. She pulls a rope over her shoulder which is attached to the sled. The boy has mass 19 kg, the sled has mass 1.4 kg, the girl is 1.2 m high at the shoulder and the rope is 2.4 m long. You may assume that the coefficient of kinetic friction μk between the sled and the snow is 0.1.

(a) Draw a diagram of the forces acting on the sled, assuming it travels at a constant speed. What are the forces acting on the girl?
(b) What is the tension in the rope?
(c) The girl changes the position she is carrying the rope, dragging the rope now at a height of 0.6 m. Does she have to increase or decrease the force with which she pulls on the rope to maintain a constant speed?
(d) The girl is now pulling the sled over grass for which $\mu_k = 0.5$. Plot a graph of the tension versus the angle between the rope and the ground, and explain what you see.

Answer to (a)

The forces are shown in Figure 15.2.

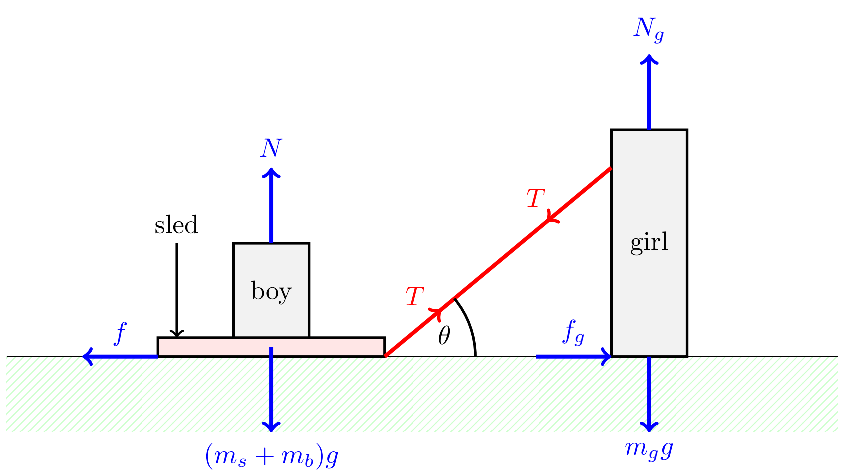

Figure 15.2: Forces acting on the girl, and sled plus boy.

The sled is acted on by

i) the tension T in the rope at $\theta = 30^\circ$ to the horizontal (assuming the sled is at negligible height, as the rope is twice as long
ii) kinetic friction f acting backwards horizontal,
iii) the weight of the sled and the weight of the boy, $(m_s + m_b)g$, acting downwards, and
iv) a normal reaction N upwards.

Since the sled moves at constant velocity, the sum of these forces should be zero.
The girl is acted on by:

i) the tension in the rope, T (at -30° to the horizontal),
ii) her weight $m_g g$ downwards,
iii) a normal force N_g upwards and
iv) a forwards static friction force, f_g.

These should cancel out because there is no net acceleration.

Answer to (b)

The forces balance, so taking components for the sled, we have the following. Horizontally,

$$T\cos\theta = f = N\mu_k \tag{15.3}$$

and vertically,

$$T\sin\theta + N = (m_s + m_b)g \tag{15.4}$$

where the angle $\theta = 30^\circ$. We can substitute $N = T\mu_k^{-1}$ from Equation 15.3 in Equation 15.4 and rearrange to get

$$T = \frac{(m_s + m_b)g}{\sin\theta + \mu_k^{-1}\cos\theta}$$

Substituting in the numbers gives

$$T = 21.9 \text{ N}$$

Answer to (c)

If the new angle of the rope is θ', we now have $\sin\theta' = 1/4$ and $\cos\theta' = \sqrt{15}/4$. Since, in the denominator of the equation we found in the previous part,

$$\frac{1}{4} + 10\frac{\sqrt{15}}{4} = 9.9 > 9.2 = \frac{1}{2} + 10\frac{\sqrt{3}}{2}$$

The new tension is smaller than the previous tension. Physically, this can be explained because at the shallower angle more of the tension contributes to the horizontal component, balancing the friction; this has a greater effect than the increase in friction due to the increased normal force.

Answer to (d)

The tension initially decreases when we start to increase θ, as there is more benefit from increasing the component of T that is lifting the sledge and reducing the friction. As θ continues to increase, T then increases for the reason given in part (c) - we start paying the price for reducing the forward component of the tension and have to pull harder. This is shown in Figure 15.3.

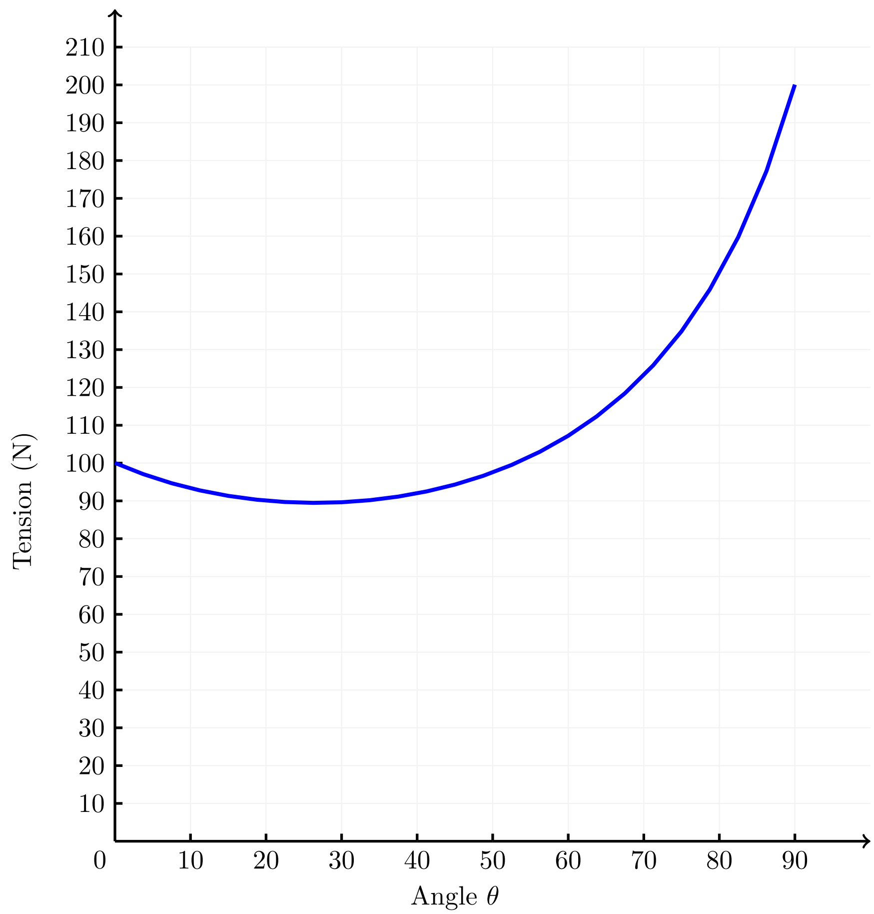

Figure 15.3: Tension in rope as a function of angle, θ, above the horizontal.

15.2 Part 2 - Problem solving question

Summative (i.e., counts 3.75% towards final grade), please submit PDF individually

The problem solving question will be provided during the workshop session.