# Mathematical Physics PHYS23020

Zoë Leinhardt

2024-11-28

# Table of contents

# Preface

This text covers the lectures for Part A of Mathematical Physics PHYS23020. If you notice any errors please email me zoe.leinhardt@bristol.ac.uk.

# Part I

# Course Notes

# 1 Introduction

Welcome to second year Mathematical Physics. This unit is divided into two sections Part A - Vector Calculus and Part B - Linear Algebra and Fourier Series.

## 1.1 Course logistics

You should familiarize yourself with the blackboard page for this unit. This is the main source of information for this unit. The lecture and problems/example class schedule should appear in your personal timetable.

Lectures will be delivered in person three times a week along with one problems class a week starting in week 2.

## 1.2 Textbooks

There are several textbooks that you may find useful. For the first week of lectures the most relevant is the first chapter of Griffiths Introduction to Electrodynamics. For the next four weeks Mathematical methods in the physical sciences by Mary Boas and Mathematical methods for physics and engineering by Riley, Hobson, and Bence will be very helpful. All of these texts are listed in the reading list and are available digitally and physically in the University Library.

# 2 Review of Vectors

Consider this situation: Walk 4 miles north and then 3 miles east (see Figure 2.1). How far have you walked?

7 miles - but you are not 7 miles from your starting point.

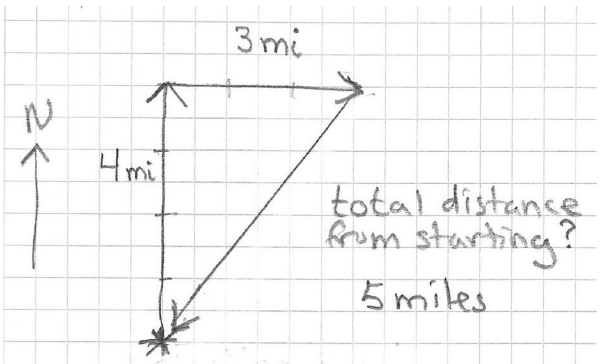



Figure 2.1: Walking Path

These quantities do not add in the standard way that we think of as addition because the displacements have direction as well as magnitude (length). Both the direction and the magnitude need to be taken into account When the quantities are combined. We also need a word for these quantities that have both magnitude and direction **vectors**.

> **Question**
>
> What are some examples of vector quantities that you often encounter in physics?

> **Answer**
>
> Velocity, acceleration, momentum, force…

However, not all quantities are vectors. Some quantities that we encounter in physics do not have direction. These quantities are **scalars**.

**Question**

What are some examples of scalar quantities that you often encounter in physics?

**Answer**

Mass, charge, density, temperature.

Vectors are identified in a number of ways - they can have a line under them or an arrow above or they many be in bold face: $\underline{A}$ or $\vec{A}$ or $\mathbf{A}$.

Scalars are not bold or can be specifically identified as the magnitude of a vector: $|A|$ or $A$.

A negative sign in front of a vector, $-\vec{A}$, means that the vector points in a direction opposite to $\vec{A}$ but has the same magnitude (see Figure 2.2).

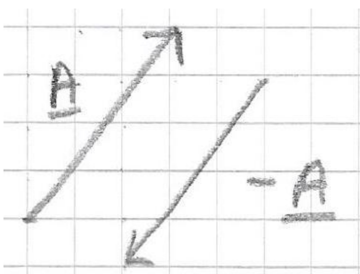



Figure 2.2: Negative vector

## 2.1 Vector Operations

Vectors do not have location - a displacement of 10 mi north from Bristol is represented by the same vector as a displacement 10 miles north from New York City. On a diagram you can slide the arrow around as long as it has the same length and direction that it did originally.

1. Addition of two vectors:

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$

3 miles east then 4 miles north gets you to the same place as 4 miles north and then 3 miles east.

- Vector addition is associative:

$$(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C}).$$

- To subtract a vector you add its opposite:

$$\underline{A} - \underline{B} = \underline{A} + (-\underline{B})$$

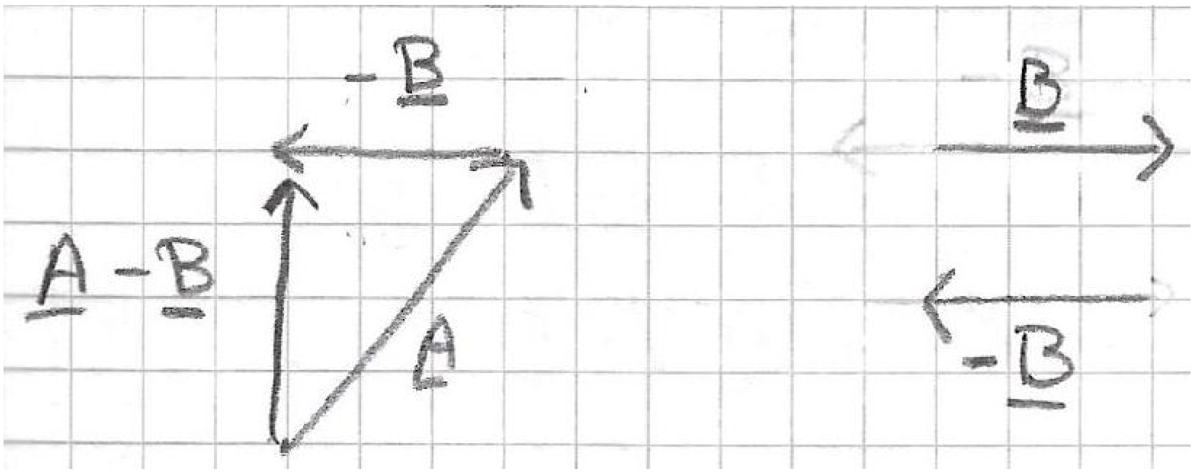



Figure 2.3: Subtracting vectors

2. Multiplication of a vector by a scalar:

$$a(\underline{A} + \underline{B}) = a\underline{A} + a\underline{B}$$

   Multiplication of a vector by a positive scalar multiplies the magnitude but leaves the direction. If the scalar is negative the direction of the vector is reversed.

3. Dot product of two vectors:

$$\underline{A} \cdot \underline{B} = AB \cos\theta,$$

   where $\theta$ is the angle between $\underline{A}$ and $\underline{B}$ when they are placed tail to tail.

- The dot product is a scalar (also called the scalar product).
- The dot product is commutative,

$$\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}.$$

- The dot product is distributive,

$$\underline{A} \cdot (\underline{B} + \underline{C}) = \underline{A} \cdot \underline{B} + \underline{A} \cdot \underline{C}.$$

**Question**

What happens if $\underline{A}$ is parallel to $\underline{B}$?

**Answer**

$$\begin{aligned}
\underline{A} \cdot \underline{B} &= AB\cos\theta \\
\theta &= 0^\circ \\
\cos\theta = \cos 0^\circ &= 1 \\
\therefore \quad A \cdot \underline{B} &= AB.
\end{aligned}$$

**Question**

What happens to the dot product if $\underline{A}$ is perpendicular to $\underline{B}$?

**Answer**

$$\begin{aligned}
&\underline{A} \cdot \underline{B} = AB\cos\theta \\
&\text{ If } A \perp B \text{ then } \theta = 90^\circ \\
&\cos 90^\circ = 0 \\
&\therefore \underline{A} \cdot \underline{B} = 0.
\end{aligned}$$

**Question**

Let $\underline{C} = \underline{A} - \underline{B}$. Calculate $\underline{C} \cdot \underline{C}$.

**Answer**

$$\begin{aligned}
\underline{C} \cdot \underline{C} &= (\underline{A} - \underline{B}) \cdot (\underline{A} - \underline{B}) \\
&= \underline{A} \cdot A - \underline{A} \cdot \underline{B} - \underline{B} \cdot \underline{A} + \underline{B} \cdot \underline{B} \\
&= A^2 + B^2 - 2\underline{A} \cdot \underline{B} \\
\underline{C} - \underline{C} &= A^2 + B^2 - 2AB\cos\theta \\
\underline{C} \cdot \underline{C} &= A^2 + B^2 - 2AB\cos\theta
\end{aligned}$$

This is the Law of Cosines

4. Cross Product of two vectors

$$\underline{A} \times \underline{B} = AB\sin\theta\,\hat{n},$$

where $\hat{n}$ is the unit vector (vector of length 1) pointing perpendicular to the plane defined by vectors $\underline{A}$ and $\underline{B}$. Note - the direction of $\hat{n}$ is ambiguous because there are two directions for this plane. It is resolved using right hand rule. Fingers point in direction of first vector and curl toward the second (smaller of two possible angles). The thumb then pants in direction of $\hat{n}$.

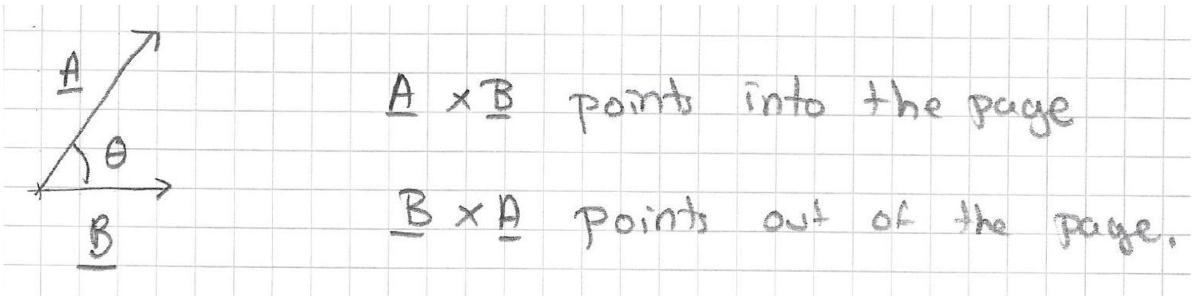



Figure 2.4: Cross product directions

- $\underline{A} \times \underline{B}$ is a vector.
- The cross product is also called the vector product.
- The cross product is distributive:

$$\underline{A} \times (\underline{B} + \underline{C}) = \underline{A} \times \underline{B} + \underline{A} \times \underline{C}.$$

- The cross produce is not commutative,

$$\underline{B} \times \underline{A} = -\underline{A} \times \underline{B}.$$

- Geometrically $\underline{A} \times \underline{B}$ is the area of the parallelogram formed by $\underline{A}$ and $\underline{B}$.

**Question**

What is the cross product of two parallel vectors?

**Answer**

$\underline{A} \times \underline{A} = 0.$

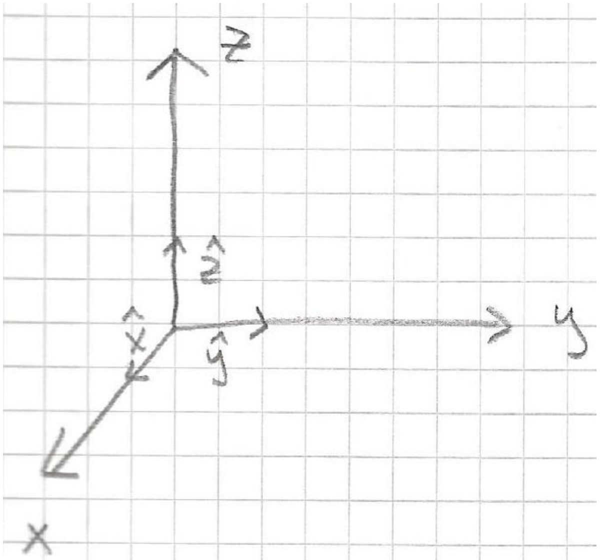



Figure 2.5: Cartesian coordinates

## 2.2 Vector Algebra

We have defined the vector function in abstract form but it is also possible to set up coordinates and work with vector components. Lets consider Cartesian coordinates - set up unit vectors $\hat{x}, \hat{y}, \hat{z}$ parallel to $x, y, z$ axes.

$$\underline{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$A_x, A_y, A_z$ are projections of $\underline{A}$ along the respective coordinate axis.

Now lets return to the four vector functions that we covered earlier.

- Vector addition:

$$\begin{aligned}\underline{A} + \underline{B} &= \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) + \left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right) \\ &= (A_x + B_x)\,\hat{x} + \left(A_y + B_y\right)\hat{y} + (A_z + B_z)\,\hat{z}\end{aligned}$$

  To add vectors in this form add components.

- Scalar multiplication:

$$a\underline{A} = aA_x\hat{x} + aA_y\hat{y} + a_z\hat{z}$$

  To multiply by a scalar multiply each component by the scalar.

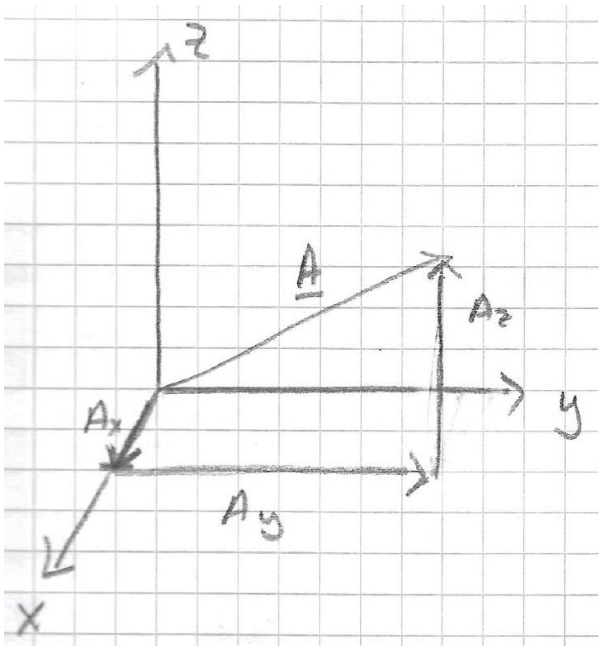



Figure 2.6: Vector A in components

- Dot product:

$$
\begin{aligned}
\underline{A} \cdot \underline{B} &= \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \cdot \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right). \\
&= A_x B_x + A_y B_y + A_z B_z
\end{aligned}
$$

  Note that since $\hat{x}, \hat{y}, \hat{z}$ are all mutually $\perp$ $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$ and $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$. Thus, to calculate dot product multiply like components and add.

**Question**

Calculate $\underline{A} \cdot \underline{A}$.

**Answer**

$$ $\underline{A}$ $\underline{A}$=A_{x}^{{2}+A_{y}}{2}+A_{z}^{2}

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

magnitude of $A$.

- Cross product:

$$\underline{A} \times \underline{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \times \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right).$$

You can multiply it all out and it will give you:

$$= \left(A_y B_z - A_z B_y\right)\hat{x} + \left(A_z B_x - A_x B_z\right)\hat{y} + \left(A_x B_y - A_y B_x\right)\hat{z}.$$

Because $\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$ and

$$\left.\begin{array}{l} \hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z} \\ \hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x} \\ \hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y} \end{array}\right\} \text{using right hand rule}$$

But there is an easier way to remember this:

$$A \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\left(A_y B_z - A_z B_y\right) - \hat{y}\left(A_x B_z - A_z\right) + \hat{z}\left(A_x B_y - A_y B_x\right)$$

**Question**

Find the angle between the face diagonals of a cube.

**Answer**

$$\underline{A} = 1\hat{x} + 0\hat{y} + 1\hat{z}$$
$$\underline{B} = 0\hat{x} + 1\hat{y} + 1\hat{z}$$
$$\underline{A}\cdot\underline{B} = 1\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1$$

also

$$\begin{aligned} & A\cdot\underline{B} = AB\cos\theta \\ & A = \sqrt{1^2+0^2+1^2} = \sqrt{2} \\ & B = \sqrt{O^2+1^2+1^2} = \sqrt{2} \\ & A\cdot B = \sqrt{2}\cdot\sqrt{2}\cos\theta \\ & \;= 2\cos\theta \\ & 1 = 2\cos\theta \\ & \theta = \cos^{-1}(1/2) \\ & \theta = 60^\circ. \end{aligned}$$

# 3 Differential Calculus

Suppose that we have a function of one variable $f(x)$

Question

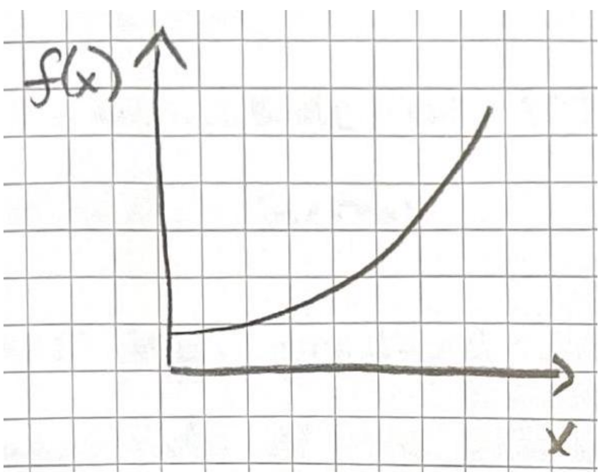
Q. What does the derivative $\frac{df}{dx}$ tell us?

Answer

Answer: It tells us how rapidly $f(x)$ varies when we change $x$ by a small amount $dx$.

$$df = \left(\frac{df}{dx}\right) dx$$

$\frac{df}{dx}$ is the slope of the graph of $f$ versus $x$




$\frac{df}{dx}$ increases as we move away from the origin.

So what happens when a function depends on more than one variable? Lets take temperature in a room - $T(x, y, z)$ ?

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$dT$ describes mathematically how T varies when we chance all variables $x, y$, and $z$ a little bit $dx, dy, dz$.

## 3.1 Gradient

We can rewrite the above as a dot product:

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$dT = \nabla T \cdot d\underline{l}$$
$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$\nabla T$ is called the gradient of $T$ and is a vector quantity. Like any other vector $\nabla T$ has both magnitude and direction.

$$dT = \nabla T \cdot d\underline{l} = |\nabla T||d\underline{l}|\cos\theta$$

where $\theta$ is the angle between $\nabla$ and $dl$.

- $\nabla T$ points in the direction of maximum increase of the function $T$.
- The magnitude of $\nabla T$ is the slope along the maximal increase.

**Question**

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

**Answer**

$$\nabla r = \frac{\partial}{\partial x}r\hat{x} + \frac{\partial}{\partial y}r\hat{y} + \frac{\partial}{\partial z}r\hat{z}$$
$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} \cdot 2x$$
$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} \cdot 2y$$
$$\frac{\partial r}{\partial z} = \frac{1}{2}\left(x^2 + y^2 + z^3\right)^{-1/2} \cdot 2z$$
$$r = \left(x^2 + y^2 + z^2\right)^{1/2} \Rightarrow \frac{1}{r} = \left(x^2 + y^2 + z^2\right)^{-1/2}$$
$$\therefore \nabla r = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \frac{\hat{r}}{|r|} = \hat{r}$$

We call $\nabla$ del and it is a vector operator. Del acts somewhat like a normal vector and can "act" in three ways

- it can multiply a scalar. $\Rightarrow$ gradient $\nabla T$
- it can be dotted with a vector $\nabla \cdot \underline{A} \Rightarrow$ we call this divergence
- it can be "crossed" with a vector $\nabla \times \underline{A} \Rightarrow$ this is called curl.

## 3.2 Divergence

$$\begin{aligned}\nabla \cdot \underline{v} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot (v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) \\ &= \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z\end{aligned}$$

The divergence of a vector is a scalar. It represents the spreading out of a vector from the point in question.

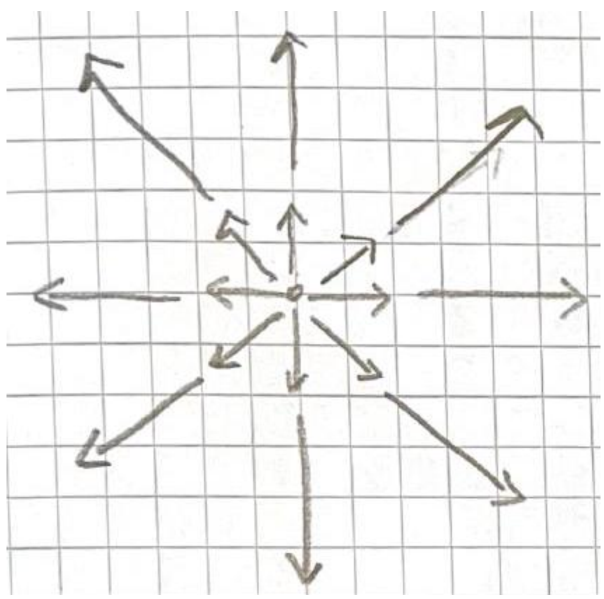


Figure 3.1: Possitive Divergence

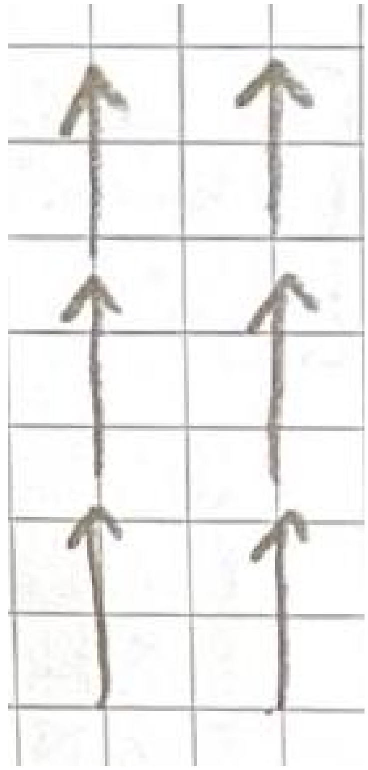


Figure 3.2: Zero divergence

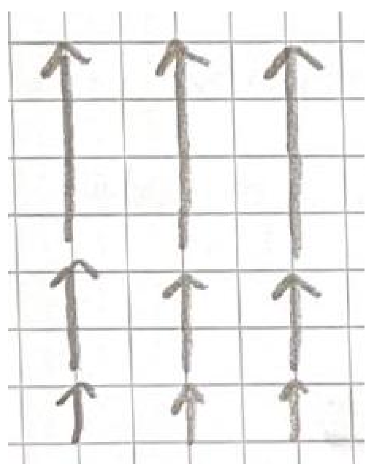


Figure 3.3: What about this one?

**Question**

$$v_a = x\hat{x} + y\hat{y} + z\hat{z}, v_b = \hat{z}, v_c = z\hat{z}$$

**Answer**

a) $\underline{\nabla} \cdot \underline{v}_a = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$ pos. divergence
b) $\underline{\nabla} \cdot \underline{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0$
c) $\underline{\nabla} \cdot \underline{v}_c = \frac{\partial(0)}{\partial x} + \frac{\partial}{\partial y}(0) + \frac{\partial z}{\partial z} = 1$ pos. divergence

## 3.3 The Curl

$$\begin{aligned}
\underline{\nabla} \times \underline{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\
&= \hat{x}\left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right] - \hat{y}\left[\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right] \\
&+ \hat{z}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]
\end{aligned}$$

- The curl of a vector is a vector

Geometric interpretation is how much a vector curls around the paint in question.

Previous figures → have 0 curl above have non-zero curl.

**Question**

$\mathbf{v}_a = -y\hat{x} + x\hat{y}, \quad \mathbf{v}_b = x\hat{y}$
Calculate curl: $\nabla \times \mathbf{v}_a$

**Answer**

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$$
\begin{aligned}
&= \hat{x}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right) - \hat{y}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right) \\
&+ \hat{z}\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right) \\
&= 0\hat{x} - 0\hat{y} + 2\hat{z}
\end{aligned}
$$

**Question**

Calculate curl: $\underline{\nabla} \times \mathbf{v}_b$

**Answer**

$$
\underline{\nabla} \times \mathbf{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix}
$$

$$
\begin{aligned}
&= \hat{x}(\partial/\partial y(0) - \partial\partial\partial z(x)) - \hat{y}(\partial/\partial x(0) - \partial\partial z(0)) + \hat{z}(\partial\partial x(x) - \partial/\partial y(0)) \\
&= 1\hat{z}.
\end{aligned}
$$

## 3.4 The Laplacian

One can also take the divergence of a gradient of a scalar field: $\nabla \cdot \nabla\phi$ or $\nabla^2\phi$ (the Laplacian of $\phi$).

$$
\begin{aligned}
\nabla^2\phi &= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial\phi}{\partial z} \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}
\end{aligned}
$$

# 4 Line, Surface, and Volume Integrals

In electrodynamics you will encounter several different kinds of integrals such as line (path), surface (flux), and volume.

## 4.1 Line integrals

$$\int_a^b \underline{v} \cdot d\underline{l}$$

where $\underline{v}$ is a vector function, $d\underline{l}$ is an infinitesimal displacement vector and the integral is calculated a long a specific path $P$ from a point $a$ to a point $b$.

If the path forms a closed loop meaning the end point is the same as the starting point $a = b$ this is shown as a circle on the integral:

$$\oint_a^b \underline{v} \cdot d\underline{l}.$$

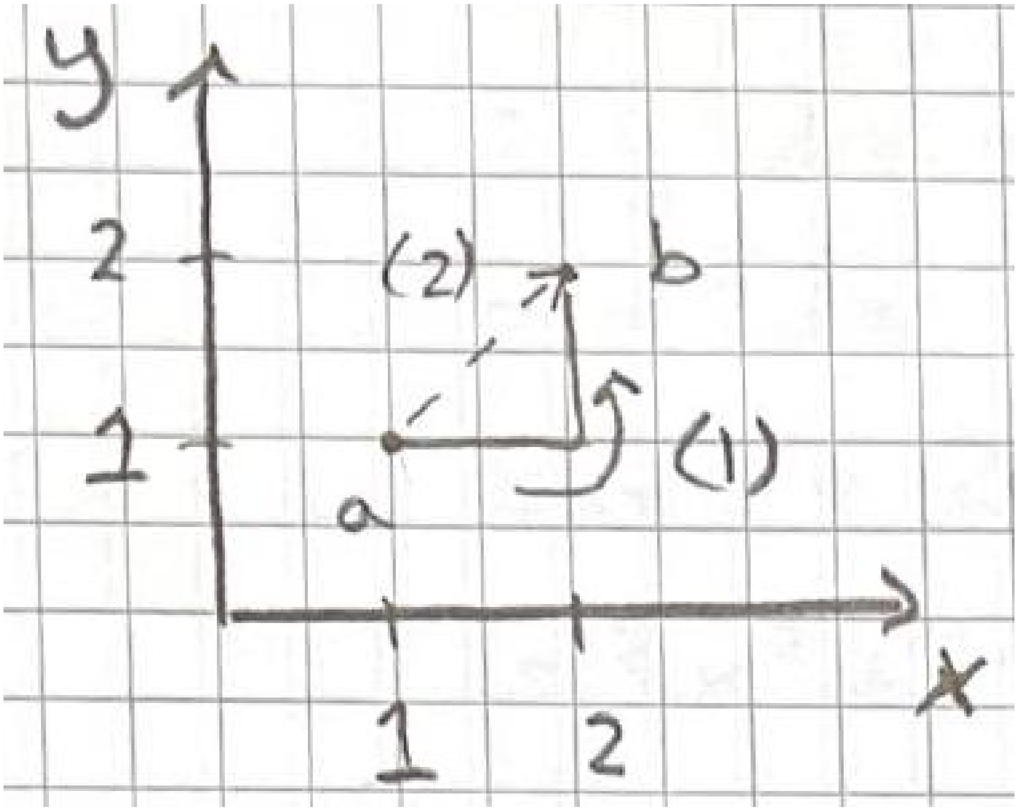
In order to calculate the path integral take the dot product of $\underline{v}$ evaluated at that point with displacement $d$ d to the next point on the path.

> **Question**
>
> Think of an example of a path integral in physics?

> **Answer**
>
> Work: $W = \int \underline{E} \cdot d\underline{l}$ Work done by a force $\underline{E}$.

Note: A conservative force is a force that is independent of path.

> **Question**
>
> Calculate the line integral of the function $\mathbf{v} = y^2\hat{x} + 2x(y+1)\hat{y}$ form point $a = (1, 1, 0)$ to point $b = (2, 2, 0)$ along paths (1) and (2).




💡 Answer

$$d\underline{\ell} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Path (1)

- first part $dy = dz = 0$, $d\underline{l} = dx\,\hat{x}$ and $y = 1$.

$$\begin{aligned}\underline{v} \cdot d\underline{l} &= y^2 dx \,@\, y = 1 \\ &= dx\end{aligned}$$

$$\int \underline{v} \cdot d\underline{l} = \int_1^2 dx = 1$$

- second part of Path 1 (the vertical part $d\underline{l} = dy\,\hat{y}$ and $x = 2$.)

$$\int \underline{v} \cdot d\underline{l} = \int_1^2 2x(y+1)dy$$

$$x = 2$$

$$= \int_1^2 4(y+1)dy$$

$$= (2y^2 + 4y)\Big|_1^2$$

$$= (8+8) - 6 = 10.$$

So all together path 1: $\int \underline{v} \cdot d\underline{l} = 11$.
Path $2 : x = y \quad dx = dy$ and $dz = 0$.

$$d\underline{l} = dx\,\hat{x} + dy\,\hat{y}$$
$$\underline{v} \cdot d\underline{l} = y^2 dx + 2x(y+1)dy$$

but $dx = dy$ and $x = y$

$$= y^2 dx + 2x(y+1)dx$$
$$= x^2 dx + 2x(x+1)dx$$
$$= (x^2 + 2x^2 + 2x)\,dx$$
$$= (3x^2 + 2x)\,dx$$

$$\int \underline{v} \cdot d\underline{l} = \int_1^2 (3x^2 + 2x)\,dx = \frac{3x^3}{3} + \frac{2x^2}{2}\Big|_1^2$$
$$= x^3 + x^2\Big|_1^2$$
$$= (8+4) - 2 = 10$$
$$\text{Path } 2 = 10$$

**Question**

What is $\oint \underline{v} \cdot d\underline{1}$ for the loop that goes from $a$ to $b$ out on path (1) and back on path (2)?

**Answer**

$$\oint \underline{v} \cdot d\underline{l} = 11 - 10 = 1$$

## 4.2 Surface Integrals

$$\int_S \underline{v} \cdot d\underline{a}$$

where $\underline{v}$ is a vector function and $d\underline{a}$ is a little bit of area with direction perpendicular to the surface.

Similarly to the path integral $\oint \underline{v} \cdot d\underline{a}$ indicates a closed surface.

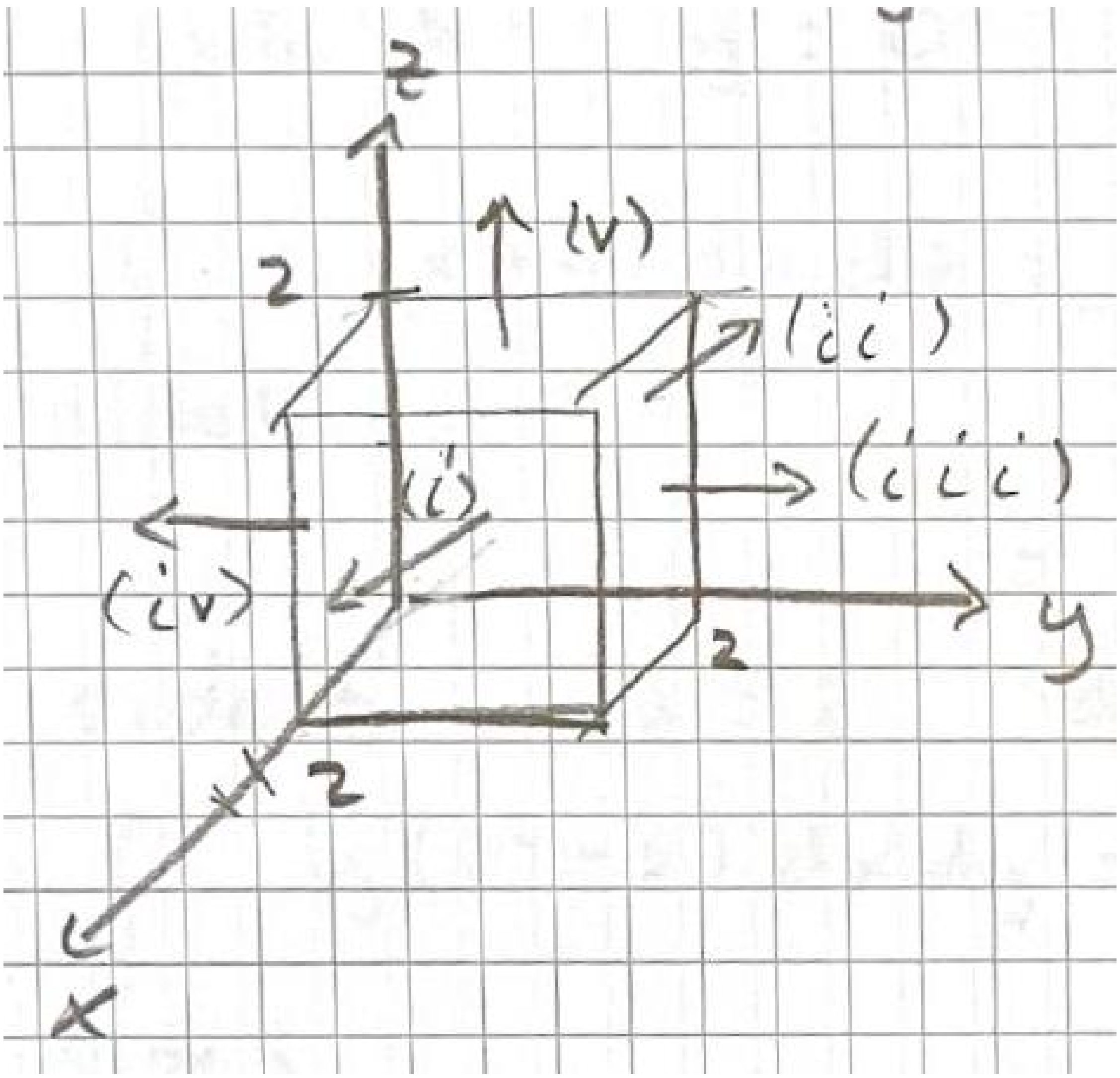
Generally outward is positive for a surface but if the surface is open this is arbitrary. If $\underline{v}$ describes flow then $\int \underline{v} \cdot d\underline{a}$ is the total mas through the surface per unit time (or flux).

**Question**

Calculate the surface integral of

$$v = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$$

over five sides of a cubical box (excluding the bottom).




Answer

Let do one side at a time:

For side (i) $x = 2$ and $d\underline{a} = dydz\hat{x}$

$$\int_{\text{side 1}} v \cdot d\underline{a} = \int_0^2 \int_0^2 2xz\, dy\, dz$$
$$= \int_0^2 dy \int_0^2 4z dz$$
$$= \left. y\right|_0^2 \cdot \left. 2z^2\right|_0^2$$
$$= 2 \cdot 8 = 16$$

Side 2 $\quad x = 0 \quad d\underline{a} = -dy\, dz\, \hat{x}$

$$\int_{\text{side 2}} \underline{v} \cdot d\underline{a} = \int -2xz\, dy\, dz = 0$$

Side 3 $\quad y = 2 \quad d\underline{a} = dx\, dz\, \hat{y}$

$$\int_{\text{side 3}} (x+2)\, dx\, dz = \int_0^2 dz \int_0^2 (x+2) dx$$
$$= \left. z\right|_0^2 \cdot \left. \left( \frac{x^2}{2} + 2x \right) \right|_0^2$$
$$= 2 \cdot 6 = 12$$

Side 4 $\quad y = 0 \quad d\underline{a} = -dx\, dz\, \hat{y}$

$$\int_{\text{side 4}} -(x+2) dx dz = -\int_0^2 dz \int_0^2 (x+2) dx$$
$$= -12$$

Side 5 $\quad z = 2 \quad d\underline{a} = dx\, dy\, \hat{z}$

$$\int_{\text{Side 5}} y\left(z^2 - 3\right) dx dy = \int_0^2 dx \int_0^2 y\, dy$$
$$= 2 \cdot \left. \frac{y^2}{2} \right|_0^2$$
$$= 4$$
$$\therefore \int_S \underline{v} \cdot d\underline{a} = 16 + 0 + 12 - 12 + 4 = 20.$$

## 4.3 Volume Intugrals

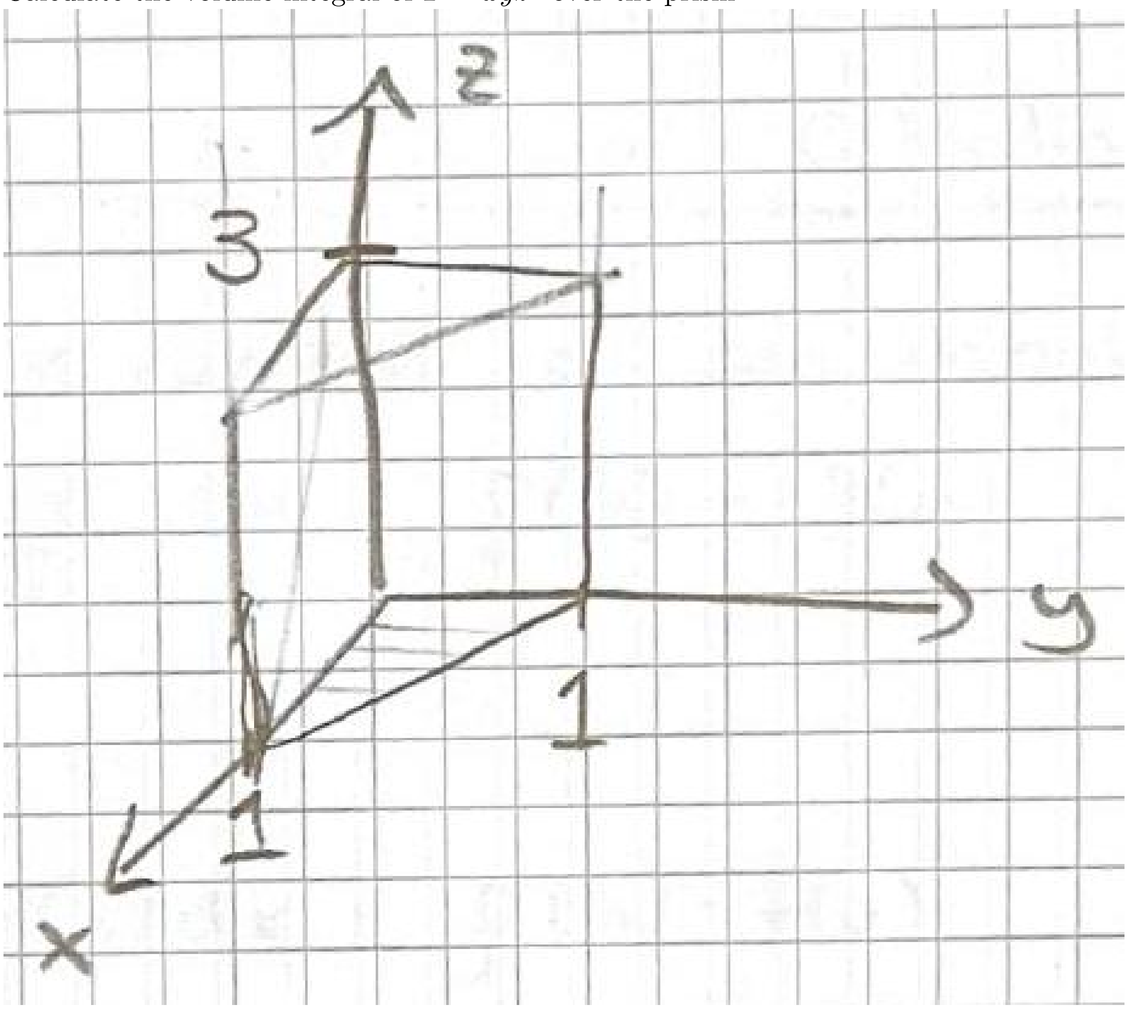
$$\int_V T d\tau$$

Where $T$ is a scalar function and $d\tau$ is a small volume element.

$$d\tau = dxdydz$$

If $T$ is the density of something that might vary from point to point the volume integral would give the total mass.

> **Question**
>
> Calculate the volume integral of $T = xyz^2$ over the prism



Answer

$$
\begin{aligned}
\int_V T d\tau &= \int_0 xyz^2 dxdydz \\
&= \int_0^3 \int_0^1 \int_0^{1-y} xyz^2 dxdydz \\
&= \int_0^3 \int_0^1 \frac{x^2}{2} yz^2 \bigg|_0^{1-y} dydz \\
&= \int_0^3 z^2 dz \int_0^1 \frac{(1-y)^2}{2} y\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{z^3}{3}\bigg|_0^3 \cdot \int_0^1 y \frac{(1-2y+y^2)}{2} dy \\
&= \frac{9}{2} \int_0^1 (y - 2y^2 + y^3)\, dy \\
&= \frac{9}{2} \left( \frac{y^2}{2} + \frac{2}{3} y^3 + \frac{y^4}{4} \right) \bigg|_0^1 \\
&= \frac{9}{2} \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\
&= \frac{9}{2} \left( \frac{6 - 8 + 3}{12} \right) \\
&= \frac{9}{2} \cdot \frac{1}{12} = \frac{3}{8}
\end{aligned}
$$

# 5 Fundamental Theorems

## 5.1 Fundamental Theorem of Calculus

$f(x)$ is a function of one variable

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

or

$$\int_a^b F(x) dx = f(b) - f(a),$$

where $F(x) = \frac{df}{dx}$. This tells you how to integrate $F(x)$ - find a function $f(x)$ with a derivate equal to $F(x)$.

## 5.2 Fundamental Theorem of Gradients

For $T(x, y, z)$ a scalar function and $dT = \nabla T \cdot d\underline{l}$

$$\int_a^b \nabla T \cdot d\ell = T(b) - T(a).$$

In other words the line integral of the gradient is given by the value of the function at its boundaries.

Note: Gradients are special - the line integrals associated with them are path independent.

> **Question**
>
> Lets check the Fundamental Theorem of Gradients assuming $T = xy^2$ point $a = (0, 0, 0)$ and $b = (2, 1, 0)$.

Answer

We need to pick a path even though gradients are special and path independent. So lets take the path from point$a$ to point $b$ in two parts first horizonally along the x-axis (from $(0,0,0) \to (2,0,0)$) and then vertically up to point $b$ ($\to (2,1,0)$).

(1) Out along $x$-axis, $d\underline{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

$$\begin{aligned}
&\int_i \nabla T \cdot d\underline{l} \\
&d\underline{l} = dx\hat{x} \\
&\nabla T = y^2\hat{x} + 2xy\hat{y}, \quad y = 0 \\
&\nabla T \cdot dl = 0 \\
&\int_i \nabla T \cdot dl = 0
\end{aligned}$$

(2) Now lets calculate the left hand side of the theorem for the second half of the path:

$$\begin{aligned}
&\int_{ii} \nabla T \cdot d\underline{l}, \quad x = 2 \\
&\nabla T \cdot d\underline{l} = 4ydy \\
&\int_0^1 4ydy = 2y^2\Big|_0^1 = 2
\end{aligned}$$

Thus the entire integral is $\int_a^b \nabla T \cdot dl = 2$. Is this consistent with the fundamental theorem of gradients? Yes' be cave $T(b) - T(a) = 2 - 0 = 2$.
Can check with another path.

(3) Lets take another path - the straight line from the origin to $(1,2,0)$

$$\begin{aligned}
y = 1/2\,x, dy = 1/2\,dx \quad \nabla T \cdot d\underline{l} &= y^2dx + 2xydy \\
&= \frac{1}{4}x^2dx + \frac{x^2}{2}dx \\
&= \frac{3}{4}x^2dx
\end{aligned}$$

$$\int_{\text{iii}}^2 \nabla \cdot d\underline{l} = \int_0^2 \frac{3}{4}x^2dx = \frac{1}{4}x^3\Big|_0^2 = 2$$

## 5.3 Fundamental Theorem of Divergence

$$\int_V (\nabla \cdot \underline{v}) dT = \oint_S \underline{v} \cdot d\underline{a}$$

This is saying that the integral of the derivative (divergence) over a region (volume) is equal to the value at of the function at the boundary (at the bounding surface of the volume). This is also called Gauss's Theorem $\Rightarrow$ super useful in electrodynamics.

The divergence represents the "spreading out" so if $\underline{v}$ represents the flow of incompressible fluid then the right hand side is the flux through the surface

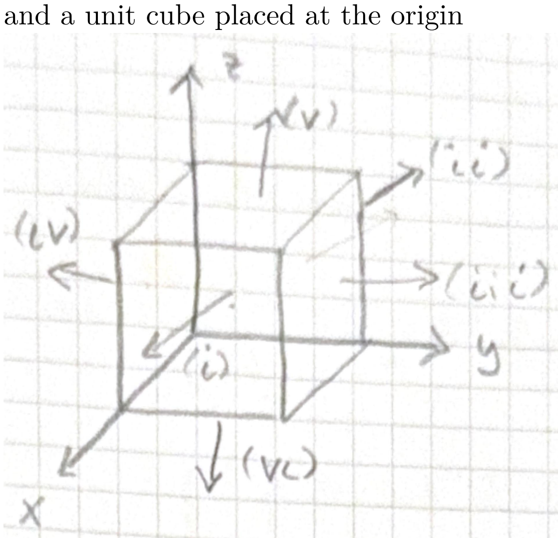
$$\int \text{faucets within the volume} = \oint \begin{array}{c}\text{flow out through}\\ \text{the surface}\end{array}$$

**Question**

Check the divergence theorem

$$\underline{v} = y^2 \hat{x} + \left(2xy + z^2\right) \hat{y} + 2yz\hat{z}$$

and a unit cube placed at the origin



Answer

$$\begin{aligned}
&\nabla \cdot \underline{v} = 0 + 2x + 2y \\
&\int_V 2(x+y)d\tau = \int_0^1 \int_0^1 \int_0^1 2(x+y)dxdydz \\
&= 2\int_0^1 \int_0^1 \left(\frac{x^2}{2} + yx\right)\Bigg|_0^1 dydz \\
&= 2\int_0^1 \int_0^1 (1/2 + y)dydz \\
&= 2\int_0^1 dz\,(1/2y + 1/2y^2)\Big|_0^1 \\
&= 2(1/2 + 1/2) = 2.
\end{aligned}$$

Left side of the divergence theorem. $\oint_S \underline{v} \cdot d\underline{a}$ (right side) $\Rightarrow$ consider each side:
Side 1: $x = 1$ and $d\underline{a} = dydz\hat{x}$

$$\underline{v} \cdot d\underline{a} = y^2 dydz$$

$$\int_{\text{side 1}} \underline{v} \cdot d\underline{a} = \int_0^1 \int_0^1 y^2 dydz = \int_0^1 \frac{y^3}{3}\Bigg|_0^1 dz$$

$$= \frac{1}{3}$$

Side 2: $x = 0 \quad da = -dydz\hat{x}$

$$\int_{\text{side 2}} y \cdot d\underline{a} = -\int_0^1 \int_0^1 y^2 dydz = -\frac{1}{3}.$$

Side 3: $y = 1 \quad d\underline{a} = dxdz\hat{y}$

$$\begin{aligned}
\int_{\text{side 3}} \underline{v} \cdot d\underline{a} &= \int_0^1 \int_0^1 2xy + z^2 dxdz \\
&= \int_0^1 \int_0^1 \left(2x + z^2\right) dxdz \\
&= \int_0^1 \left(x^2 + xz^2\right)\Big|_0^1 dz \\
&= \int_0^1 1 + z^2 dz \\
&= z + \frac{z^3}{3}\Big|_0^1 \\
&= \frac{4}{3}.
\end{aligned}$$

Side 4: $y = 0 \quad d\underline{a} = -dxdz\hat{y}$

$$\begin{aligned}
\int_{\text{side 4}} \underline{v} \cdot d\underline{a} &= -\int_0^1 \int_0^1 z^2 dydz \\
&= -\int_0^1 z^2 dz \\
&= -\frac{1}{3}.
\end{aligned}$$

Side 5: $z = 1 \quad d\underline{a} = dxdy\hat{z}$

$$\begin{aligned}
\int_{side5} \underline{v} \cdot d\underline{a} &= \int_0^1 \int_0^1 2ydxdy \\
&= \int_0^1 2ydy = 1.
\end{aligned}$$

side 6: $z = 0 \quad d\underline{a} = -dxdy\hat{z}$

$$\begin{aligned}
\int_{side6} \underline{v} \cdot d\underline{a} &= -\int_0^1 \int_0^1 0dxdy = 0 \\
\therefore \oint_S \underline{v} \cdot d\underline{a} &= \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 \\
&= 2
\end{aligned}$$

## 5.4 Fundamental Theorem of Curls

$$\int_S (\nabla \times \underline{v}) \cdot d\underline{a} = \oint_P \underline{v} \cdot d\underline{l}$$

The integral of a derivative (curl) over a region (patch of surface) equals the value at the boundary (path). This is also called Stokes Theorem.

The left side depends only on the boundary line not the surface used.

Note: For a closed surface

$$\oint_S (\nabla \times \underline{v}) \cdot d\underline{a} = 0.$$

**Question**

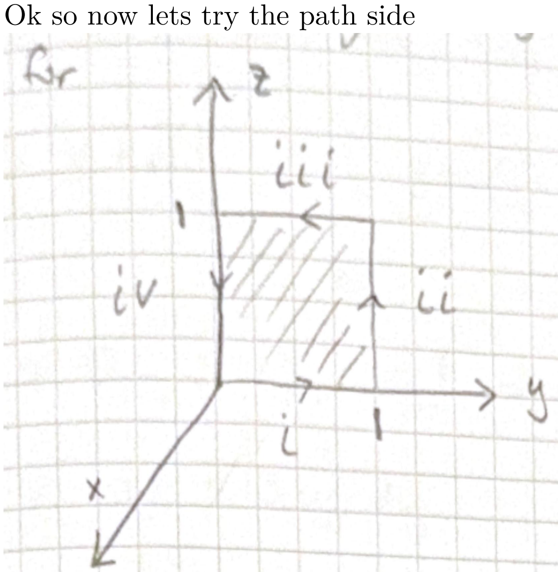
Check Stokes theorem - $\underline{v} = (2xz + y^2)\,\hat{y} + (4yz^2)\,\hat{z}$.

Answer

$$
\begin{aligned}
\nabla \times \underline{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} \\
&= \hat{x}\left(\frac{\partial 4yz^2}{\partial y} - \frac{\partial}{\partial z}\left(2xz + 3y^2\right)\right) \\
&- \hat{y}\left(\frac{\partial}{\partial x}4yz^2 - \frac{\partial 0}{\partial z}\right) \\
&+ \hat{z}\left(\frac{\partial}{\partial x}\left(2xz + 3y^2\right) - \frac{\partial}{\partial y}(0)\right) \\
&= \hat{x}\left(4z^2 - 2x\right) - \hat{y}(0) + \hat{z}(2x) \\
&= \left(4z^2 - 2x\right)\hat{x} + 2x\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
d\underline{a} &= dydz\hat{x}, x = 0, (\nabla \times \underline{v}) \cdot d\underline{a} = \left(4z^2 - 2x\right)\hat{x} \cdot dydz\hat{x} \\
&= \left(4z^2 - 2x\right) dydz \\
\int \nabla \times \underline{v} \cdot d\underline{a} &= \int_0^1 \int_0^1 \left(4z^2 - 2x\right) dydz \\
&= \int_0^1 \int_0^1 4z^2 dydz \\
&= \int_0^1 4z^2 dz = 4/3.
\end{aligned}
$$

Ok so now lets try the path side



$$\oint \underline{v} \cdot d\underline{l} = \int_{i} \underline{v} \cdot d\underline{l} + \int_{ii} \underline{v} \cdot d\underline{l} + \int_{iii} \underline{v} \cdot d\underline{l} + \int_{iv} \underline{v} \cdot d\underline{l}$$

Part i

$$x = 0, z = 0, \quad d\underline{l} = dy\hat{y}$$

$$\underline{v} \cdot d\underline{l} = 3y^2 dy \qquad \int_i \underline{v} \cdot d\underline{l} = \int_0^1 3y^2 dy = 1$$

Part ii

$$y = 1, x = 0, d\underline{l} = dz\hat{z}$$

$$\underline{v} \cdot d\underline{l} = 4yz^2 dz = 4z^2 dz$$

$$\int_{ii} \underline{v} \cdot d\underline{l} = \int_0^1 4z^2 dz = \frac{4}{3}.$$

Part iii

$$x = 0, z = 1, d\underline{l} = -dy\hat{y}$$

$$\underline{v} \cdot d\underline{l} = -\left(2xz + 3y^2\right) dy = -3y^2 dy$$

$$\int_{iii} \underline{v} \cdot d\underline{l} = \int_1^0 3y^2 dy = y^3\Big|_1^0 = -1$$

Part iv

$$x = 0, y = 0 \quad d\underline{l} = -dz\hat{z}$$

$$\underline{v} \cdot d\underline{l} = -4yz^2 dz = 0.$$

$$\int_{iv} \underline{v} \cdot d\underline{l} = \int_1^0 0 dz = 0$$

$$\oint v \cdot d\underline{l} = 1 + 4/3 - 1 + 0 = 4/3$$

# 6 Differentiation of vectors

Lets consider the derivative of $\mathbf{a}(u)$ with respect to $u$. The derivative of a vector is defined in a similar way to the derivative of a scaler.

$$\Delta \mathbf{a} = \mathbf{a}(u + \Delta u) - \mathbf{a}(u)$$

$$\frac{d\mathbf{a}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{a}(u + \Delta u) - \mathbf{a}(u)}{\Delta u}$$

$\frac{d\mathbf{a}}{du}$ is a vector. In Cartesian coordinates

$$\text{if } \mathbf{a} = a_x \hat{\imath} + a_y \hat{\jmath} + a_z \hat{k}$$

$$\frac{d\mathbf{a}}{du} = \frac{da_x}{du}\hat{\imath} + \frac{da_y}{du}\hat{\jmath} + \frac{da_z}{du}\hat{k}.$$

Lets find the velocity of a particle

$$\mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

The direction of $\mathbf{v}(t)$ is tangent to the path $\mathbf{r}(t)$ and $|\mathbf{v}(t)|$ is the speed of the particle.

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$$

**Question**

$\mathbf{r}(t) = 2t^2\hat{\imath} + (3t - 2)\hat{\jmath} + \left(3t^2 - 1\right)\hat{k}$ Find the speed of the particle at $t = 1$ and the component of its acceleration in the direction $\mathbf{s} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$.

💡 Answer

$$\frac{d\mathbf{r}}{dt} = 4t\hat{\imath} + 3\hat{\jmath} + 6t\hat{k}$$
$$v(1) = \sqrt{4^2 + 9 + 36}$$
$$= \sqrt{61}$$

Now lets find the acceleration: $\mathbf{a} = 4\hat{\imath} + 6\hat{k}$ $\mathbf{a}$ is independent of time.
To find the component of $\mathbf{a}$ in the direction of $\mathbf{s}$ we need to find the unit vector $\hat{s}$ and project $\mathbf{a}$ in that direction:

$$\mathbf{a} \cdot \hat{s} = (4\hat{\imath} + 6\hat{k}) \cdot (\hat{\imath} + 2\hat{\imath} + \hat{k})/\sqrt{6} = (4 + 0 + 6)/\sqrt{6} = 10/\sqrt{6}$$

## 6.1 Plane Polar Coordinates

What if another coordinate system is more appropriate?

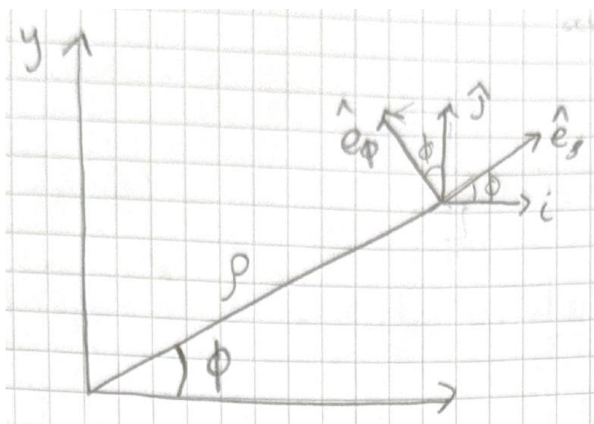



Figure 6.1: Plane Polar Coordinates

Lets start by writing the polar coordinates in terms of Cartesian $\hat{\imath} + \hat{\jmath}$

$$
\begin{aligned}
&\hat{e}_\rho = \cos\phi\hat{\imath} + \sin\phi\hat{\jmath} \\
&\hat{e}_\phi = -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\
&\frac{d\hat{e}_\rho}{dt} = -\sin\phi\frac{d\phi}{dt}\hat{\imath} + \cos\phi\frac{d\phi}{dt}\hat{\jmath} \\
&= \dot{\phi}\hat{e}_\phi \\
&\frac{d\hat{e}_\rho}{dt} = -\cos\phi\frac{d\phi}{dt}\hat{\imath} - \sin\phi\frac{d\phi}{dt}\hat{\jmath} \\
&\frac{d\hat{e}_\rho}{dt} = -\dot{\phi}\hat{e}_\rho
\end{aligned}
$$

The overdot is standard notation for a time derivative.

**Question**

$\mathbf{r}(t) = \rho(t)\hat{e}_\rho$. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ in these coordinates.

**Answer**

$$
\begin{aligned}
&\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{\rho}\hat{e}_\rho + \rho\dot{\hat{e}}_\rho = \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi \\
&\mathbf{a}(t) = \frac{d}{dt}\left(\dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi\right) \\
&= \ddot{\rho}\hat{e}_\rho + \dot{\rho}\dot{\hat{e}}_\rho + \dot{\rho}\dot{\phi}\hat{e}_\phi + \rho\ddot{\phi}\hat{e}_\phi + \rho\dot{\phi}\dot{\hat{e}}_\phi \\
&= \ddot{\rho}\hat{e}_\rho + \dot{\rho}\dot{\phi}\hat{e}_\phi + \dot{\rho}\dot{\phi}\hat{e}_\phi + \rho\ddot{\phi}\hat{e}_\phi - \rho\dot{\phi}^2\hat{e}_\rho \\
&= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{e}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{e}_\phi
\end{aligned}
$$

## 6.2 Differentiation of composite vector expressions

Lets consider a scalar $\phi$ and a vectors $\mathbf{a}$ and $\mathbf{b}$.

$$
\begin{aligned}
&\frac{d}{du}(\phi\mathbf{a}) = \phi\frac{d\mathbf{a}}{du} + \frac{d\phi}{du}\mathbf{a} \\
&\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\cdot\mathbf{b} \\
&\frac{d}{du}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\times\frac{\mathbf{b}}{du}
\end{aligned}
$$

**Question**

A particle of mass $m$ with position vector $\mathbf{r}$ relative to some origin $O$ experiences a force $\mathbf{F}$ which produces a torque (moment) $\mathbf{T} = \mathbf{v} \times \mathbf{F}$ about $O$. The angular momentum is given by $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$. Show that the time rate of change angular momentum $\frac{d\mathbf{L}}{dt} = \mathbf{T}$

**Answer**

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{d}{dt}(r \times mv) \\
&= \left( r \times m\frac{dv}{dt} + \frac{dr}{dt} \times m\underline{v} \right) \\
&= v \times m\underline{v} + r \times m\frac{dv}{dt} \\
&= 0 + r \times E \\
{}^{\prime}{}_{r}\frac{dL}{dt} &= I
\end{aligned}
$$

# 7 Integration of vectors

The integral has the same nature as the integrand (vector or scalar)

$$
\begin{aligned}
&\int \mathbf{a}(u) du = \mathbf{A}(u) + (\mathbf{b}) - \text{ where } \mathbf{b} \text{ is a constant vector} \\
&\int_{u_1}^{u_2} a(u) du = \mathbf{A}(u_2) - \mathbf{A}(u_1)
\end{aligned}
$$

Question

A small particle mass $m$ orbits a much bigger mas $M$ located at the origin.

$$
m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\hat{r}
$$

Show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt}$ is a constant of motion.

Answer

First step lets take the vector product of force equation

$$
\mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \left(-\frac{GMm}{r^2}\right)\hat{r}
$$

$m$ is a constant divide both sides

$$
\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \left(-\frac{GM}{r^2}\right)\hat{r}
$$

- $\frac{GM}{r^2}$ is a constant

$$
\begin{aligned}
&\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = -\frac{Gm}{r^2}(\mathbf{r} \times \hat{r}) \\
&\mathbf{r} \times \hat{r} = \mathbf{0}
\end{aligned}
$$

which means

$$
\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0}.
$$

Ok but have not gotten to what we need quite yet - we have a $\frac{d^2\mathbf{r}}{dt^2}$ which is one derivative higher

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt} + \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$$
$$\therefore \frac{d}{dt}\left(r\times\frac{dr}{dt}\right) = \underline{0}$$

a vector crossed with itself is $\mathbf{0}$ and we already showed that the second term is zero.

integrate the above and we find $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$ = constant.

# 8 Gradient of a scalar field

Lets consider the problem of calculating the rate of change of a scalar $\phi$ in some particular direction. For an infinitesimal vector displacement $d\mathbf{r}$ forming its scalar product we get

$$\begin{aligned}\nabla\phi \cdot d\mathbf{r} &= \left(\frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz) \\ &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\end{aligned}$$

## 8.1 Partial Derivative

Ok lets pause for a second and look at exactly what a partial derivative is.

Lets take a function of two variables $f(x, y)$ We can define a derivative for $f(x, y)$ in $x$ by saying that it is the derisive of $f(x, y)$ when holding $y$ constant (we could do the same with respect to $y$ holding $x$ constant). We write this $\frac{\partial f}{\partial x}$ the partial derivative of $f(x, y)$ with respect to $x$. Similarly $\frac{\partial f}{\partial y}$. Formally this is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and similarly for $\frac{\partial f}{\partial y}$. You may also see partials written as $\left(\frac{\partial f}{\partial x}\right)_y$ would indicate $y$ is held constant.

## 8.2 Total differential and total derivative

Suppose we make small changes in $x$ and $y$

$$\begin{aligned}\Delta f &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= \left[\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\right]\Delta x + \left[\frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y}\right]\Delta y \\ \Delta f &\approx \frac{\partial f(x, y)}{\partial x}\Delta x + \frac{\partial f(x, y)}{\partial y}\Delta y\end{aligned}$$

$\Delta x + \Delta y \to 0$ we get the 'total differential'

$$df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

this can he extended to

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \cdots \frac{\partial f}{\partial x_n}dx_n$$

Returning back to our original work

$$\nabla\phi \cdot d\mathbf{r} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

the right hand side is the total differential in $\phi$.

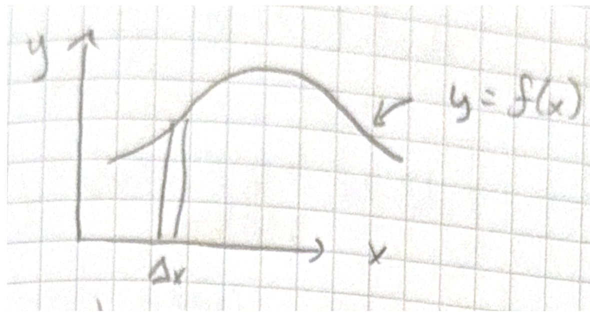
Now lets consider $x$ and $y$ are functions of $u$

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ \frac{df}{du} &= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{dy} \\ \frac{df}{du} &= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{dx_i}{du} \end{aligned}$$

Chain Rule for partial differentiation.

# 9 Multiple Integrals

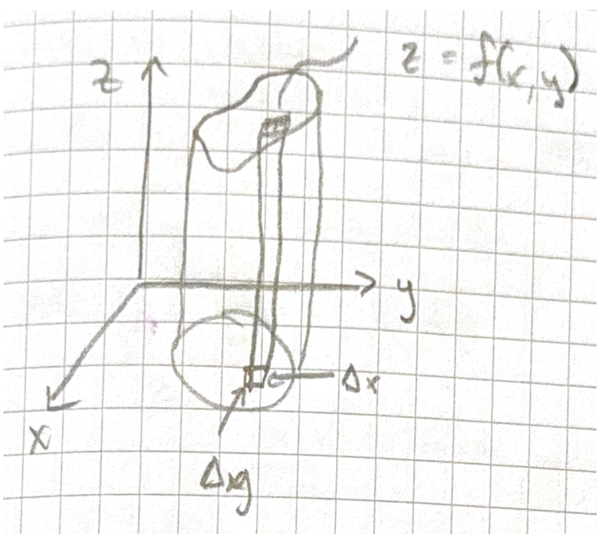
## 9.1 Double Integrals




$\int_a^b y dx = \int_a^b f(x) dx \Rightarrow$ area "under the curve"

Define $\int_a^b f(x) dx$ as the limit of the sum of the areas of rectangles.

What about three dimensional regions?




Create columns and add them up to find the volume. The column shown above has a cross section of $\Delta A = \Delta x \Delta y$.

Question

Find volume of the solid bounded by the coordinate planes ($xz$, $yz$, $xy$) and the planes $z = 1 + y$ and $-2x + y = 2$. (Example from Boas book - section on multiple integrals)

Answer

volume is integration (adding up) columns of height $z$ and cros-sectional area $\Delta A = \Delta x \Delta y$

$$V = \iint z dx dy = \iint (1 + y) dx dy$$

Now decide which way to integrate first - integrate over y first holding x constant or integrate over x first and hold y constant (we could do either). I have chosen the former - in the xy plane build up a slice at a specific x value. So integrage over y first creating a slice of thickness dx. Now we need the limits for y. We start at y=0 and then reach the line y=2x+2. The value of the max y depends on the x value so the upper limit must be in terms of x.

$$\begin{aligned}
V &= \int_{x=0}^{x=1} \int_{y=0}^{y=2x+2} (1 + y) dy dx \\
&= \int_{x=0}^{x=1} \left( y + \frac{y^2}{2} \Big|_0^{2-2x} \right) dx \\
&= \int_0^1 2 - 2x + \frac{(2-2x)^2}{2} dx \\
&= \int_0^1 2 - 2x + \frac{4 - 8x + 4x^2}{2} dx \\
&= \int_0^1 4 - 6x + 2x^2 dx \\
&= 4x - 3x^2 + \frac{2}{3}x^3 \Big|_0^1 \\
&= 4 - 3 + \frac{2}{3} = \frac{5}{3}
\end{aligned}$$

Alternatively we could have added up the columns into slices keeping $y$ constant and we would have gotten the same answer.

$$\int_{y=0}^{y=2}\int_{x=0}^{x=2-y/2}(1+y)dxdy$$

$$\int_0^2 (1+y)x\Big|_{x=0}^{x=1-y/2} dy$$

$$\int_0^2 (1+y)\cdot(1-y/2)dy$$

$$\int_0^2 1-\frac{y}{2}+y-\frac{y^2}{2}dy$$

$$\int_0^2 1+\frac{y}{2}-\frac{y^2}{2}dy$$

$$2+\frac{y^2}{4}-\frac{y^3}{6}\Big|_0^2$$

$$3-\frac{4}{3}=\frac{5}{3}$$

The volume calculation above is an example of an iterated integral - holding a variable constant calculating one integral (inner most) then integrating over the variable that you just held constant. We did that example in both options x then y and y then x but often one way is easier than the other.

In the special case of rectangle (both $x$ and $y$ are constant)

$$f(x,y)=g(x)h(y)$$

then $\int_{x=1}^{b}\int_{y=c}^{d} g(x)h(y)dydx$

$$\left(\int_0^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Can use double integral to find more than just volume.

**Question**

Find mass of rectangular plate bounded by $x+0, x=2 \quad y=0, y=1$ if its density (mass/area), is $f(x,y)=xy$.

Answer

$$
\begin{aligned}
&\Delta A = \Delta x A \Delta y \\
&\Delta M = f(x,y)\Delta x \Delta y \\
&M = \iint_A f(x,y)dxdy \\
&\int_{y=0}^{y=2}\int_{x=0}^{x=1} xydxdy \\
&\int_0^1 ydy \int_0^2 xdx \\
&\left.\frac{y^2}{2}\right|_0^1 \cdot \left.\frac{x^2}{2}\right|_0^2 = 1
\end{aligned}
$$

## 9.2 Triple Integrals

We could have used a triple integral in the first problem - to create the column in the $z$ direction.

Question

Find $V$ of solid in previous problem using a triple integral.

Answer

$$V = \iiint dxdydz$$

Now figure out order - we want to create column in z first so integrate in z first holding both x and y constant then lets hold x constant and integrate over y.

$$
\begin{aligned}
V &= \iiint dxdydz \\
&= \int_0^1 \left( \int_0^{2-2x} \left[ \int_0^{1+y} dz \right] dy \right) dz \\
&= \int_{x=0}^1 \int_{y=0}^{2-2x} (1+y)dydx = 5/3
\end{aligned}
$$

# 10 Directional Derivative

Lets say you are on a hillside and you want to know in what direction does the hill slope downward most steeply from this point? (This would be the direction you would slide if you lost your footing) - this would be the direction straight down.

Lets say we move a small distance $\Delta s$ the vertical change will be $\Delta z$ (positive, negative, or zero) meaning $\frac{dz}{ds}$ depends upon duration (it is a directional derivative).

The direction of steepest slope is the direction in which $\frac{dz}{ds}$ has its largest value.

Lets take a scalar field $\phi(x, y, z)$. To find the directional derivative of $\phi$ at a particular point in a particular direction we need to find $\frac{d\phi}{ds}$ the rate of change of $\phi$ with distance at a given point $x_0, y_0, z_0$ and in a given direction ($\mathbf{s}$).

Lets define $\mathbf{u} = \hat{\imath}a + \hat{\jmath}b + \hat{k}c$ as a unit vector in a given direction ($\mathbf{s}$ in this case).

$$
\begin{aligned}
&(x, y, z) - (x_0, y_0, z_0) = \mathbf{u}s = (\hat{\imath}a + \hat{\jmath}b + \hat{k}c)s \\
&x = x_0 + as \\
&y = y_0 + bs \\
&z = z_0 + cs
\end{aligned}
$$

parametric equations - $x, y, z$ are equations of a single variable.

Lets write the full derivative:

$$
\begin{aligned}
\frac{d\phi}{ds} &= \frac{\partial \phi}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{ds} \\
\frac{d\phi}{ds} &= \frac{\partial \phi}{\partial x} \cdot a + \frac{\partial \phi}{\partial y} \cdot b + \frac{\partial \phi}{\partial z} \cdot c \\
\nabla \phi &= \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k} \\
\nabla \phi \cdot \mathbf{u} &= \frac{\partial \phi}{\partial x}a + \frac{\partial \phi}{\partial y}b + \frac{\partial \phi}{\partial z}c
\end{aligned}
$$

Thus the directional derivative is $\frac{d\phi}{ds} = \nabla \phi \cdot \mathbf{u}$.

**Question**

Find the directional derivative of $\phi = x^2y + x$ at $(1, 2, -1)$ in the direction $\mathbf{A} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

**Answer**

$$\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|}$$
$$|\mathbf{A}| = \sqrt{4 + 4 + 1} = 3$$
$$\mathbf{u} = \frac{1}{3}(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$$
$$\nabla\phi = (2xy + z)\hat{\imath} + x^2\hat{\jmath} + x\hat{k}$$

at point $(1, 2, -1)$

$$\nabla\phi(1, 2, -1) = (2 \cdot 1 \cdot 2 - 1)\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$$
$$= 3\hat{\imath} + \hat{\jmath} + \hat{k}$$
$$\frac{d\phi}{ds} = \nabla\phi \cdot \mathbf{u} = 2 - 2/3 + 1/3 = 5/3$$

$\frac{d\phi}{ds} = |\nabla\phi||\underline{u}|\cos\theta = |\nabla\phi|\cos\theta$ where $\theta$ is angle between $\nabla\phi$ and $\mathbf{u}$.

Maximum $\frac{d\phi}{ds}$ or $|\nabla\phi|$ is if $\theta = 0$ largest decrease occurs at $\theta = 180$ or $-|\nabla\phi|$.

**Question**

Lets consider the temperature in a room. The temperature follows $T = x^2 - y^2 + xyz + 273$. In which direction is the temp increasing most rapidly at $(-1, 2, 3)$?

**Answer**

$$\nabla T = (2x + yz)\hat{i} + (-2y + xz)\hat{\jmath} + xy\hat{k}$$
$$\nabla T(-1, 2, 3) = (-2 + 6)\hat{i} + (-4 - 3)\hat{\jmath} + (-2)\hat{k}$$
$$= 4\hat{i} - 7\hat{\jmath} - 2\hat{k}$$

and max rate of $\Delta$ in direction of this vectors. $\frac{dT}{ds} = |\Delta T| = \sqrt{69}$. $-\nabla T$ is rate of max decrease $\frac{dT}{ds} = -\sqrt{69}$. Heat flows in $-\Delta T$ direction.

## 10.1 Normal Derivative

Now say $\phi$ is constant at a point $P\,(x_0, y_0, z_0)$ and the direction vector $\mathbf{u}$ is tangent to $\frac{d\phi}{ds}$. If $\phi$ is constant then $\frac{d\phi}{ds} = 0$. This also means $\nabla\phi \cdot \mathbf{u} = 0$.

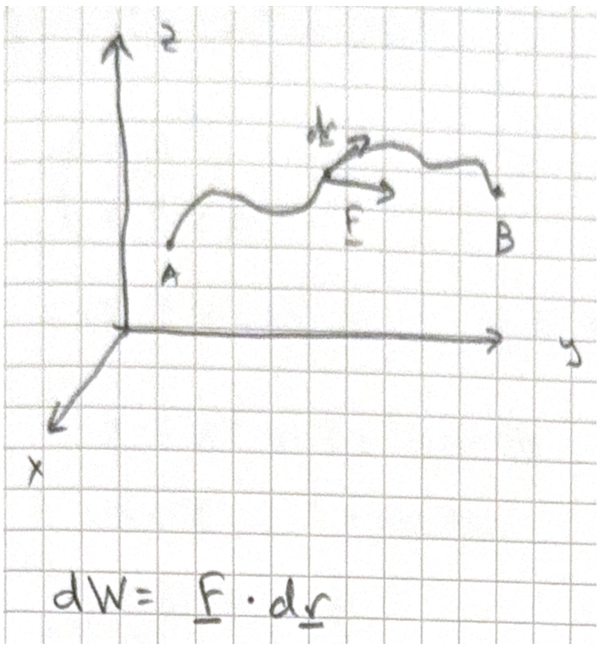
$$|\nabla\phi||\mathbf{u}|\cos\theta = 0$$
$$|\nabla\phi|\cos\theta = 0$$

$$\theta = 90^\circ$$

In this case, $\nabla\phi$ is perpendicular to the surface. Since $|\nabla\phi|$ is the value of the directional derivative in the direction normal often called the normal derivative

$$|\nabla\phi| = \frac{d\phi}{dn}.$$

# 11 Line Integrals




Lets say we have a charged particle in an electric field and we need to calculate the work needed for it to move from point $A$ to $B$ along a particular path. The coordinates $x, y, +z$ are constrained by the equation of the path. The path is one dimensional and can be written in terms of one variable. The total work is the integral over the path and can be written in terms of the one independent variable either by writing two of the coordinates in terms of the other or by writing parametric equations.

$$x = x_0 + as$$
$$y = y_0 + bs$$
$$z = z_0 + cs$$

**Question**

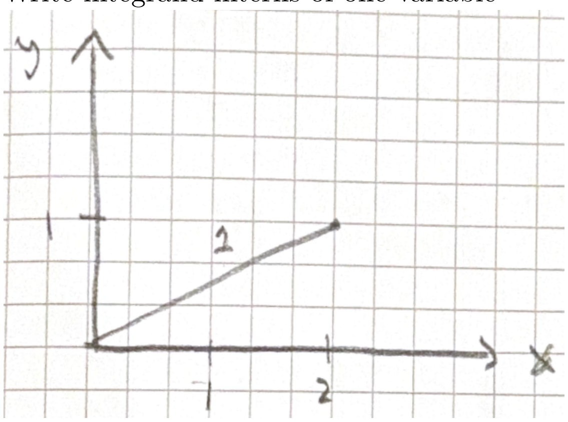
$$\mathbf{F} = xy\hat{\imath} - y^2\hat{\jmath}$$

Answer

Find work done along the path

$$
\begin{aligned}
d\mathbf{r} &= \hat{\imath}dx + \hat{\jmath}dy \\
\mathbf{F} \cdot d\mathbf{r} &= xydx - y^2dy \\
W &= \int_{\text{Path}} xydx - y^2dy
\end{aligned}
$$

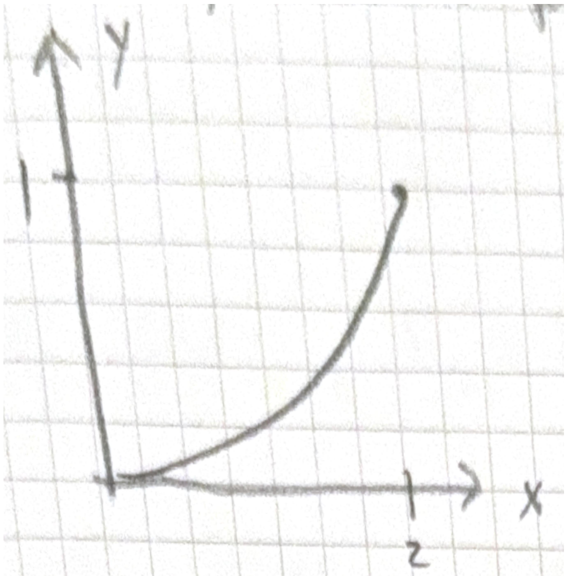
Write integrand interns of one variable




Lets start by taking path 1 a straight line

$$
\begin{aligned}
y &= 1/2x \\
dy &= 1/2dx \\
W &= \int_0^2 \left(x \cdot \frac{1}{2}x - \frac{1}{4}x^2 \cdot \frac{1}{2}\right) dx \\
&= \int_0^2 \left(\frac{1}{2}x^2 - \frac{1}{8}x^2\right) dx \\
&= \int_0^2 \frac{3}{8}x^2 dx \\
&= \frac{1}{8} \cdot x^3 \Big|_0^2 = 1
\end{aligned}
$$

Now lets consider another path - a parabola




$$
\begin{aligned}
y &= \frac{1}{4}x^2 \quad dy = \frac{1}{2}xdx \\
W_2 &= \int_0^2 x \cdot \frac{1}{4}x^2 - \frac{1}{16}x^4 \cdot \frac{1}{2}xdx \\
&= \int_0^2 \left(\frac{1}{4}x^3 - \frac{1}{32}x^5\right) dx \\
&= \frac{x^4}{16} - \frac{1}{192}x^6\Big|_0^2 \\
&= 1 = \frac{64}{192} - \frac{2}{3}
\end{aligned}
$$

The work calculated in the two paths is not the same. But there are several examples when it is (gravitational field for example). If the force field is path dependent it is a non-conservative force if it is path independent it is a conservative force.

A conservative force field means that $\int \mathbf{F} \cdot d\mathbf{r}$ is the same no matter what path is taken.

Suppose

$$
\begin{aligned}
&\mathbf{F} = \nabla W \cdot d\mathbf{r} = \hat{\imath}\frac{\partial W}{\partial x} + \hat{\jmath}\frac{\partial W}{\partial y} + \hat{k}\frac{\partial W}{\partial z} \\
&F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, F_z = \frac{\partial W}{\partial z} \\
&\frac{\partial F_x}{\partial y} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial F_y}{\partial x}
\end{aligned}
$$

similarly

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \text{ and } \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\therefore \nabla \times F = \mathbf{0}$$

Thus if $\mathbf{F} = \nabla W$ then $\nabla \times \mathbf{F} = \mathbf{0}$ and it also turns out that the reverse is true as well – if $\nabla \times \mathbf{F} = \mathbf{0}$ than there is a function $W$ for which $\mathbf{F} = \nabla W$.

Now if

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} = \nabla W \cdot dr &= \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz \\ &= dW \end{aligned}$$

and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B dW = W(B) - W(A)$$

Where $W(B)$ and $W(A)$ mean the values of the function $W$ at the end points $A \& B$ of the path. Since the integral only depends on the end pants the integration is path independent and $\mathbf{F}$ is a conservative field.

Just as we saw in problem 7 of week 3 problems the curl of the grad is always zero.

# 12 General derivation of $\nabla$ operator

Define a point in space $u, v, w$ in Cartesian it would be $(x, y, z)$ in cylindrical it is $(s, \phi, z)$ and in spherical $(r, \theta, \phi)$. The general coordinate system must be mutually orthogonal call it $\hat{u}, \hat{v}, \hat{w}$. An infinitesimal displacement: $d\mathbf{l} = fdu\hat{u} + gdv\hat{v} + hdw\hat{w}$. $f$, $g$, $h$ are functions of position characteristic of the particular coordinate system in Cartesian $f = g = h = 1$. In cylindrical $f = h = 1$ and $g = s$.

## 12.1 Gradient

If you move from point $u, v, w$ to $u + du, v + dv, w + dw$ a scalar function $t(u, v, w)$ varies by (partial differential)

$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv + \frac{\partial t}{\partial w}dw$$

We can write this as a dot product:

$$dt = \nabla t \cdot d\mathbf{l} = \nabla t_u f du + \nabla t_v g dv + \nabla t_w h dw$$
$$\text{if } \nabla t_u = \frac{1}{f}\frac{\partial t}{\partial u}, \nabla t_v = \frac{1}{g}\frac{\partial t}{\partial v}, \nabla t_w = \frac{1}{h}\frac{\partial t}{\partial w}$$

This would mean the gradient would be

$$\nabla t = \frac{1}{f}\frac{\partial t}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{w}$$

Now what would that mean for our coordinate systems? For cartesian $f = g = h = 1$ so

$$\nabla t = \frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z}$$

for cylindrical $f = h = 1$ and $g = s$

$$\nabla t = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Lets just check that this general definition satisfies the theorem ot gradients. The total change in $t$ as you go from point $a$ to $b$

$$\int_a^b dt = \int_a^b (\nabla t) \cdot d\mathbf{l}$$
$$= t(b) - t(a)$$

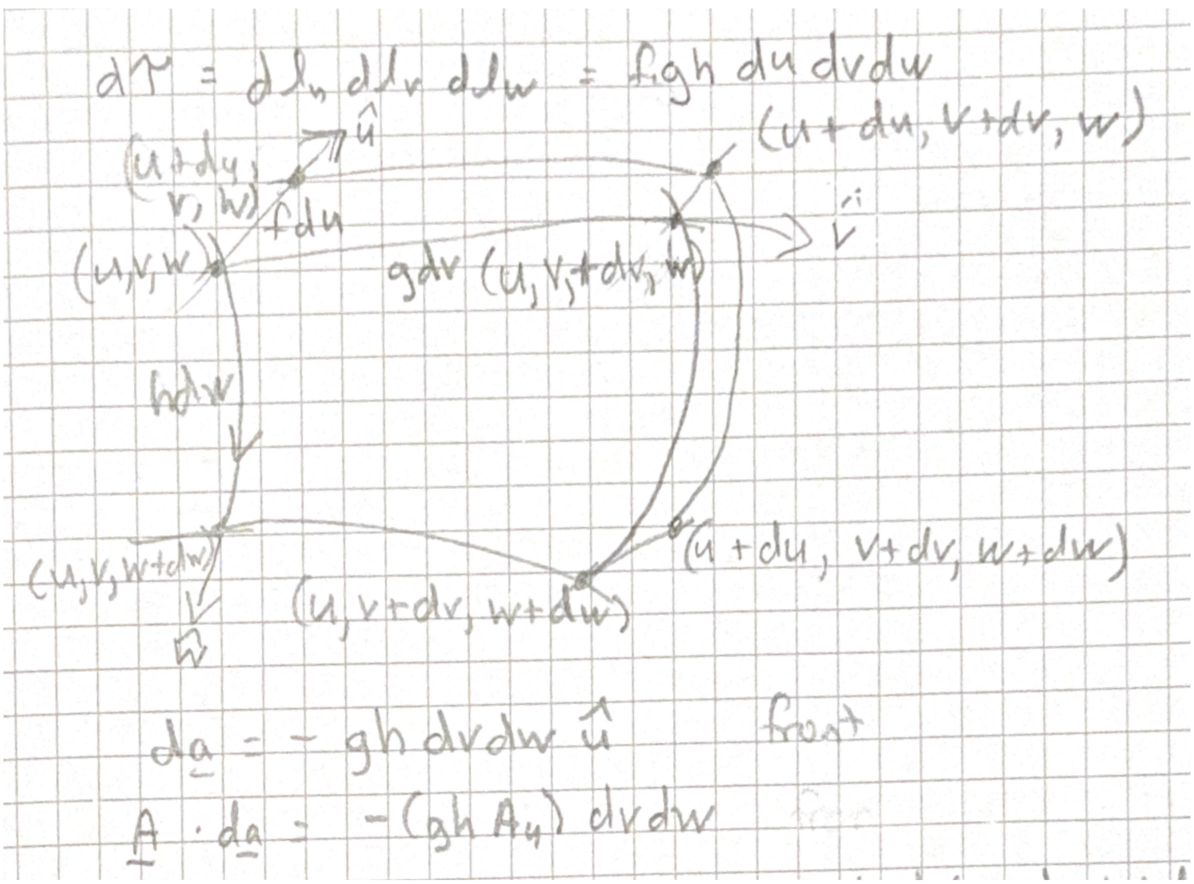
and is by definition path independent.

## 12.2 Divergence

Lets define a vector:

$$\mathbf{A}(u, v, w) = A_u \hat{u} + A_v \hat{v} + A_w \hat{w}$$

We need to evaluate $\oint \mathbf{A} \cdot d\mathbf{a}$ over a surface of infinitesimal volume generated at a point $u, v, w$




back has opposite sign and is evaluated at $u + du$

$$
\begin{aligned}
F(u+du) - F(u) &= \frac{dF}{du}du \\
\frac{\partial}{\partial u}\left(ghA_u\right)dudvdw &= \frac{1}{fgh}\frac{\partial}{\partial u}\left(ghA_u\right)d\tau
\end{aligned}
$$

left and right hand sides yield

$$
\frac{\partial}{\partial v}\left(fhA_v\right)dudvdw = \frac{1}{fgh}\frac{\partial}{\partial v}\left(fhA_v\right)d\tau
$$

and top and bottom

$$
\frac{\partial}{\partial w}\left(f_gA_w\right)dudvdw = \frac{1}{f_{gh}}\frac{\partial}{\partial w}\left(f_gA_w\right)dT
$$

Then all together

$$
\begin{aligned}
\oint \mathbf{A}\cdot d\mathbf{a} &= \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(ghA_u\right) + \frac{\partial}{\partial v}\left(fhA_v\right) + \frac{\partial}{\partial w}\left(fgA_w\right)\right]d\tau \\
\nabla\cdot\mathbf{A} &= \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(ghA_u\right) + \frac{\partial}{\partial v}\left(fhA_v\right) + \frac{\partial}{\partial w}\left(fgA_w\right)\right]\Big]
\end{aligned}
$$

Over finite $\oint \mathbf{A}\cdot d\mathbf{a} = \int(\nabla\cdot\mathbf{A})d\tau$

For cylindrical coordinates the divergence is:

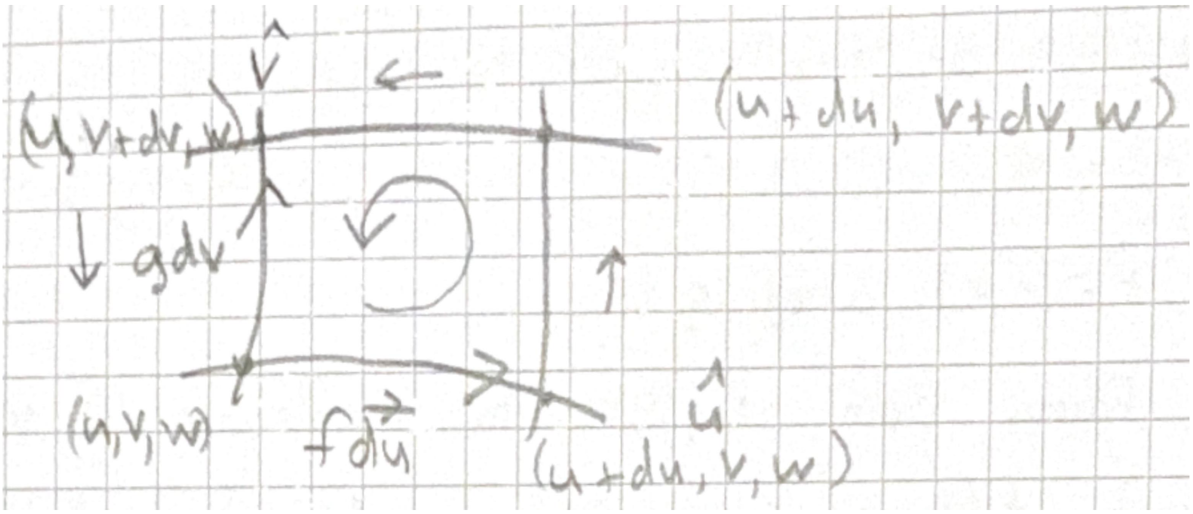
$$
\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{1}{s}\left(\frac{\partial}{\partial s}sA_s + \frac{\partial}{\partial\phi}A_\phi + \frac{\partial}{\partial z}sA_z\right) \\
\nabla\cdot\mathbf{A} &= \frac{1}{s}\frac{\partial\left(sA_s\right)}{\partial s} + \frac{1}{s}\frac{\partial}{\partial\phi}A_\phi + \frac{\partial}{\partial z}A_z
\end{aligned}
$$

## 12.3 Curl

Remember the fundamental theorem of curls:

$$
\int_S(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l}.
$$

So if we start with the right hand side the line integral $\oint \mathbf{A}\cdot d$




$\hat{w}$ out of page

$$d\mathbf{a} = fg \text{ dudv } \hat{w}$$

along bottom segment $d\mathbf{l} = f du\hat{u}$

$$\mathbf{A} \cdot d\mathbf{l} = (fA_u)\, du$$

along top leg A is evaluated at $v + dv$

$$\left(|- (fA_u)|_{v+dv} + (fA_u)|_v\right) du = -\left[\frac{\partial}{\partial v}(fA_u)\right] dudv$$

right and left hand sides give

$$\left[\frac{\partial}{\partial u}(gA_v)\right] dudv$$

total is

$$\begin{aligned}\oint A \cdot d\mathbf{a} &= \left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right] dudv \\ &= \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right] \hat{w} \cdot d\mathbf{a}\end{aligned}$$

This provides the $w$ component of curl.

If we then do the other directions (a general patch not just in the u-v plane) we will get:

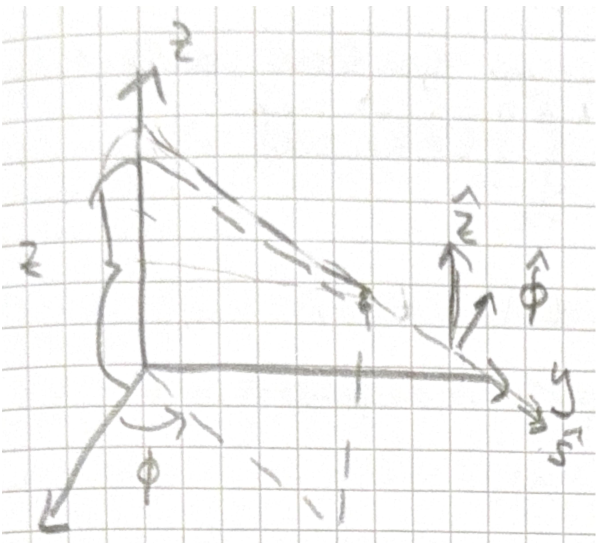
$$\begin{aligned}\nabla \times \mathbf{A} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{u} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{v} \\ + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{w}\end{aligned}$$

## 12.4 Laplacian

Derive by using divergence and gradient in general form.

$$\nabla^2 t = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\right)\frac{\partial t}{\partial u} + \frac{\partial}{\partial v}\left(\frac{fh}{g}\right)\frac{\partial t}{\partial v} + \frac{\partial}{\partial w}\left(\frac{fg}{h}\right)\frac{\partial t}{\partial w}\right]$$

# 13 Cylindrical Coordinates




$\phi$ - angle around $x$-axis 'azimithal' angle $z$ - height in $z$-axis $s$ - distance from $z$-axis

The relation to Cartesian coordinates

$$
\begin{aligned}
&x = s\cos\phi \quad y = s\sin\phi \quad z = z \\
&\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y} \\
&\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \\
&\hat{z} = \hat{z} \\
&dl_s = ds \quad dl_\phi = sd\phi \quad dl_z = dz \\
&d\mathbf{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}
\end{aligned}
$$

volume element $d\tau = sdsd\phi dz$. The range of $s$ is 0 to $\infty, \phi$ is 0 the $2\pi, z$ is $-\infty$ to $\infty$

Vector operations in cylindrical coordinates ($f = h = 1 \quad g = s$):

Gradient:

$$\nabla t_u = \frac{1}{f}\frac{\partial t}{\partial u}, \nabla t_v = \frac{1}{g}\frac{\partial t}{\partial v}, \nabla t_w = \frac{1}{h}\frac{\partial t}{\partial w}$$

$$\nabla t = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v} + (fhA_v)\frac{\partial}{\partial w}(fgA_w)\right]$$

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\begin{aligned}\nabla \times \mathbf{A} = &\frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{u}\\ &+ \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{v}\\ &+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{w}\end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{v} = &\left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi}\\ &+ \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z}\end{aligned}$$

Laplacian:

$$\nabla^2 t = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\right)\frac{\partial t}{\partial u} + \frac{\partial}{\partial v}\left(\frac{fh}{g}\right)\frac{\partial t}{\partial v} + \frac{\partial}{\partial w}\left(\frac{fg}{h}\right)\frac{\partial t}{\partial w}\right]$$

$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

**Question**

Find the divergence of the function

$$\mathbf{v} = s\left(2 + \sin^2\phi\right)\hat{s} + s\sin\phi\cos\phi\hat{\phi} + 3z\hat{z}$$

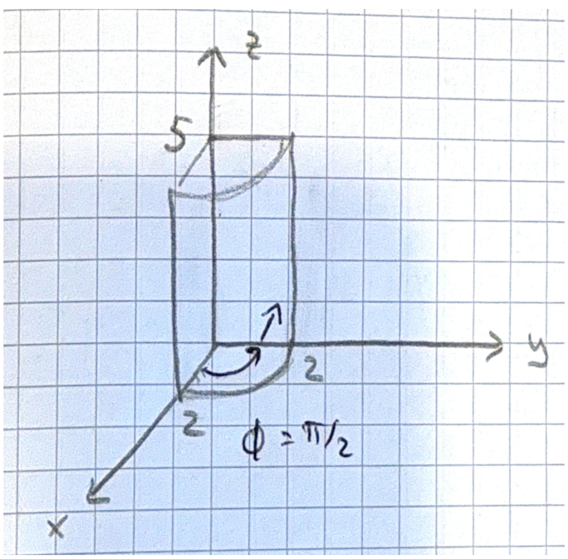
Answer

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left[s\left(2s + s\sin^2\phi\right)\right] + \frac{1}{s}\frac{\partial}{\partial \phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}3z \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(2s^2 + s^2\sin^2\phi\right) + \frac{1}{s}\cdot s\frac{\partial}{\partial \phi}(\sin\phi\cos\phi) + 3 \\
&= \frac{2}{s}\cdot 2s + 2s\sin^2\phi + \cos\phi\cos\phi - \sin\phi\sin\phi + 3 \\
&= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3 \\
&= 7 + \sin^2\phi + \cos^2\phi = 8.
\end{aligned}
$$

Question

Test the divergence theorem for this function using a quarter-cylinder ($r = 2, h = 5$).

Answer




$$\int_V (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

Left side - we have already calculated the div

$$d\tau = s\,dsd\phi dz$$

$$\int 8s\,dsd\phi dz = \int_0^5 \int_0^{\pi/2} \int_0^2 8s\,dsd\phi dz$$

All constants so can be split up

$$\begin{aligned}\int_v (\nabla \cdot \mathbf{v})\,d\tau &= 8\int_0^5 dz \int_0^{\pi/2} d\phi \int_0^2 sds \\ &= 8\left( \left. z\right|_0^5 \cdot \left.\phi\right|_0^{\pi/2} \left.\frac{s^2}{2}\right|_0^2 \right) \\ &= 8(5 \cdot \pi/2 \cdot 2) \\ \int (\nabla \cdot \mathbf{v}) d\tau &= 40\pi\end{aligned}$$

Right hand side

$$\begin{aligned}\oint_S \mathbf{v} \cdot d\mathbf{a} = &\int_{top} \mathbf{v} \cdot d\mathbf{a} + \int_{bottom} \mathbf{v} \cdot d\mathbf{a} + \int_{front} \mathbf{v} \cdot d\mathbf{a} + \int_{back} \mathbf{v} \cdot d\mathbf{a} \\ &+ \int_{left} \mathbf{v} \cdot d\mathbf{a}\end{aligned}$$

Top:

$$d_{\mathbf{a}} = sd\phi ds\hat{z}, z = 5$$

$$\mathbf{v} \cdot d\mathbf{a} = 3z \cdot sd\phi ds = 15sd\phi ds$$

$$\begin{aligned}\int_0^2 \int_0^{\pi/2} 15ssd\phi ds &= \pi/\ 2 \cdot \left.\frac{15s^2}{2}\right|_0^2 \\ &= 15\pi\end{aligned}$$

Bottom:

$$d\mathbf{a} = -sd\phi ds\hat{z}, z = 0$$

$$\iint 0 d\phi ds = 0.$$

Back:

$$da = dzds\hat{\phi}, \phi = \pi/2$$

$$\mathbf{v} \cdot d\mathbf{a} = s\sin\phi\cos\phi dsdz$$

$$\int_0^s \int_0^2 s\sin\phi\cos\phi dsdz = 0.$$

Left:

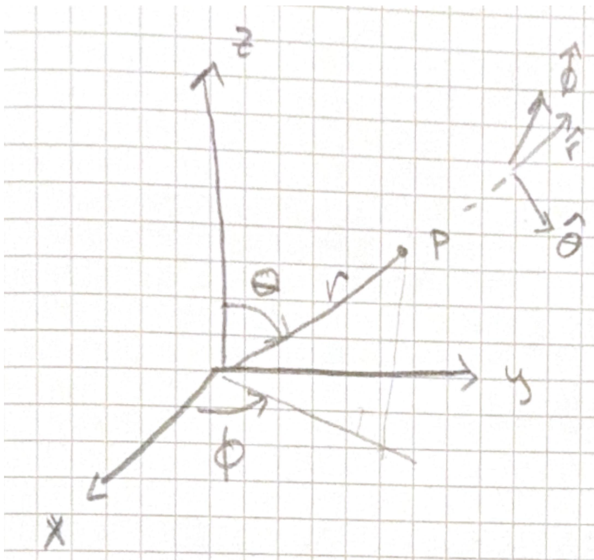
$$d\mathbf{a} = -dsdz\hat{\phi} \quad \phi = 0$$

$$\int_0^5 \int_0^2 s \sin\phi \cos\phi ds dz = 0$$

Front:

$$\begin{aligned}
d\mathbf{a} &= sd\phi dz\hat{s} \quad s = 2 \\
\mathbf{v} \cdot d\mathbf{a} &= s^2 \left(z + \sin^2\phi\right) d\phi dz \\
\int \mathbf{v} \cdot d\mathbf{a} &= \int_0^5 \int_0^{\pi/2} s^2 \left(2 + \sin^2\phi\right) d\phi dz \\
&= \int_0^5 dz \int_0^{\pi/2} 4 \left(2 + \sin^2\phi\right) d\phi \\
&= 5 \cdot 5\pi = 25\pi \\
\text{Total } &= 40\pi
\end{aligned}$$

# 14 Spherical Coordinates




$P$ is defined by $r, \theta, \phi$ in spherical coordinates. $r$ is the distance from the origin

$\theta$ is the angle down from $z$-axis

$\phi$ is the angle around $z$ - axis (same as cylindrical coords)

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ are mutually orthogonal

$$\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

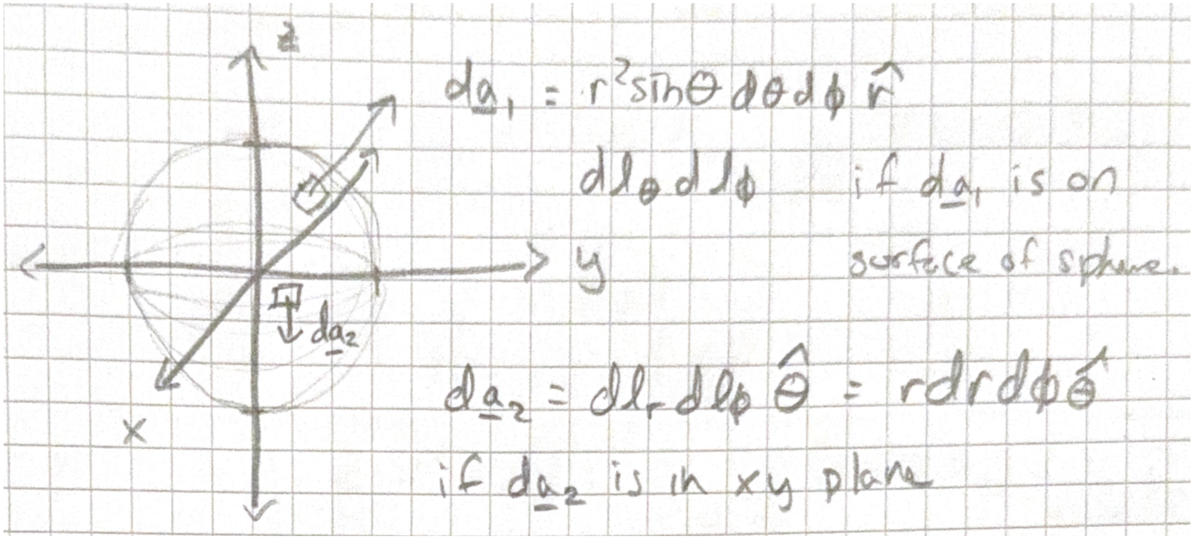
$$dl_r = dr$$
$$dl_\theta = rd\theta$$
$$dl_\phi = r\sin\theta d\phi$$

arc lengths for angles must be converted into lengths

$$d\mathbf{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$
$$d\tau = dl_r dl_\theta dl_\phi = r^2\sin\theta dr d\theta d\phi$$

r ranges from 0 to $\infty, \theta$ from 0 to $\pi$, $\phi$ from 0 to $2\pi$

Lets just think a bit about area




**Question**

Find volume of a sphere of radius $R$.

**Answer**

$$\begin{aligned}
V = \int d\tau &= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta d\phi d\theta dr \\
&= \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= \frac{r^3}{3}\Bigg|_0^R \cdot -\cos\theta\Bigg|_0^\pi \cdot \phi\Bigg|_0^{2\pi} \\
&= \frac{R^3}{3} 2 \cdot 2\pi \\
V &= \frac{4}{3}\pi R^3
\end{aligned}$$

Now lets use the general expressions to find the $\nabla$ operators for spherical coordinates.

$$\begin{aligned}
&\nabla t_u = \frac{1}{f}\frac{\partial t}{\partial u}, \nabla t_v = \frac{1}{g}\frac{\partial t}{\partial v}, \nabla t_w = \frac{1}{h}\frac{\partial t}{\partial w} \\
&d\underline{l} = f du\hat{u} + g dv\hat{v} + h dw\hat{w} \\
&f = 1, g = r, h = r\sin\theta \\
&\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}
\end{aligned}$$

Divergence:

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v} + (fhA_v)\frac{\partial}{\partial w}(fgA_w)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(rA_\phi)\right] \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(rA_\phi)
\end{aligned}$$

Curl:

$$\begin{aligned}
\nabla\times\mathbf{A} &= \frac{1}{gh}\left[\frac{\partial}{\partial v}(hA_w) - \frac{\partial}{\partial w}(gA_v)\right]\hat{u} \\
&+ \frac{1}{fh}\left[\frac{\partial}{\partial w}(fA_u) - \frac{\partial}{\partial u}(hA_w)\right]\hat{v} \\
&+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{w} \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial \theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial \phi}(rA_\theta)\right]\hat{r} \\
&+ \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(\sin\theta A_\phi)\right]\hat{\theta} \\
&+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right]\hat{\phi} \\
&= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial}{\partial \phi}(A_\theta)\right]\hat{r} \\
&+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(A_\phi)\right]\hat{\theta} \\
&+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right]\hat{\phi}
\end{aligned}$$

Laplacian:

$$\begin{aligned}
\nabla^2 t &= \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\right)\frac{\partial t}{\partial u} + \frac{\partial}{\partial v}\left(\frac{fh}{g}\right)\frac{\partial t}{\partial v} + \frac{\partial}{\partial w}\left(\frac{fg}{h}\right)\frac{\partial t}{\partial w}\right] \\
\nabla^2 t &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 t}{\partial \phi^2}\right)
\end{aligned}$$

**Question**

Compute div of

$$\underline{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$$

**Answer**

$$\begin{aligned}
\nabla\cdot\underline{v} =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \\
=& \frac{1}{r^2}\frac{\partial}{\partial r} r^3\cos\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot r\sin\theta) \\
&+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\
=& \frac{1}{r^2}3r^2\cos\theta + \frac{r}{r\sin\theta}2\sin\theta\cdot\cos\theta \\
&+ \frac{1}{r\sin\theta}r\sin\theta\cdot -\sin\phi \\
=& 3\cos\theta + 2\cos\theta - \sin\phi \\
=& 5\cos\theta - \sin\phi
\end{aligned}$$

**Question**

Now check divergence theorem for this function using an inverted hemisphere of radius $R$ as the volume and surface

**Answer**

Div theorem

$$\int_V (\nabla\cdot\underline{v})d\tau = \oint_S \underline{v}\cdot d\underline{a}$$

Left side first:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}(5\cos\theta-\sin\phi)r^2\sin\theta drd\theta d\phi$$
$$\int_0^{R}r^2dr\int_0^{\pi/2}\int_0^{2\pi}(5\cos\theta-\sin\phi)\sin\theta d\phi d\theta$$
$$\int_0^{R}\left.\frac{r^3}{3}\right|_0^R\int_0^{\pi/2}5\phi\cos\theta+\left.\cos\phi\right|_{\phi=0}^{2\pi}\sin\theta d\theta$$
$$\frac{R^3}{3}\int_0^{\pi/2}[5\cdot 2\pi\cos\theta+(1-1)]\sin\theta d\theta$$
$$\frac{R^3}{3}\int_0^{\pi/2}10\pi\cos\theta\sin\theta d\theta$$

let $u=\sin\theta$

$$du=\cos\theta d\theta$$
$$\frac{R^3}{3}\int 10\pi u du$$
$$\frac{R^3}{3}\cdot 10\pi\left.\frac{u^2}{2}\right|_{\theta=0}^{\theta=\pi/2}$$
$$\frac{R^3}{3}\cdot 10\pi\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}$$
$$\frac{R^3}{3}\cdot 10\pi\cdot\frac{1}{2}=\frac{5\pi R^3}{3}.$$

Right side: Two surfaces - hemisphere and bottom
Start with hemisphere
$\hat{r}$ component from $\underline{v}$ $\quad r=R$

$$\int\underline{v}\cdot d\underline{a}=\iint R\cos\theta\cdot R^2\sin\theta d\theta d\phi$$
$$=R^3\int_0^{2\pi}d\phi\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

$$\begin{aligned} u &= \sin\theta \\ du &= \cos\theta d\theta \end{aligned}$$

$$= R^3 \cdot 2\pi \cdot \frac{u^2}{2}\bigg|_{\theta=0}^{\theta=\pi/2}$$

$$= 2\pi R^3 \cdot \frac{\sin^2\theta}{2}\bigg|_0^{\pi/2}$$

$$= \pi R^3$$

Flat bottom

$$d\underline{a} = r\sin\theta dr d\phi\hat{\theta} \quad \theta = \pi/2$$

$$\int \underline{v}\cdot d\underline{a} = \int_0^{2\pi}\int_0^R r\sin\theta \cdot r\sin\theta dr d\phi$$

$$= 2\pi \cdot \int_0^R r^2 dr = 2\pi \cdot \frac{r^3}{3}\bigg|_0^R$$

$$= \frac{2\pi}{3}R^3 \quad \therefore \text{Total } = \frac{5\pi}{3}R^3$$

# 15 Problems

## 15.1 Week 2

Question

1. Find the gradients of:

$$f(x,y,z) = x^2 + y^3 + z^4$$
$$f(x,y,z) = x^2y^3z^4$$
$$f(x,y,z) = e^x \sin(y)\, ln(z)$$

Answer

$$2x\hat{x} + 3y^2\hat{y} + 4z^3\hat{z}$$
$$2xy^3z^4\hat{x} + 3x^2y^2z^4\hat{y} + 4x^2y^3z^3\hat{z}$$
$$e^x \sin(y)\, ln(z)\hat{x} + e^x \cos(y)\, ln(z)\hat{y} + \frac{e^x \sin(y)}{z}\hat{z}$$

Question

2. The height of a hill (in feet) is given by the function: $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$ where $y$ is the distance (in miles) north, $x$ is the distance east of the town of Trout.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point one mile north and one mile east of Trout? In what direction is the slope steepest, at that point?

Answer

Part (a)

$$\nabla h = 0 \quad \text{at summit}$$
$$\nabla h = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$$

Both $\hat{x}$ component and the $\hat{y}$ component are equal to 0 at the summit.

$$\begin{aligned}
\text{x component: } & 2y - 6x - 18 = 0 \\
\text{y component: } & 2x - 8y + 28 = 0 \\
& y = 3 \quad x = -2 \\
& \text{three miles north, two miles west of Trout.}
\end{aligned}$$

Part (b)

$$h(-2, 3) = 720 \text{ ft}$$

Part (c)

$$\nabla h(1, 1) = -220\hat{x} + 220\hat{y}$$
$$|\nabla h| = 220\sqrt{2} \text{ northwest}$$

Question

3. Calculate the divergence of the following vector functions:

$$\begin{aligned}
\mathbf{v_a} &= x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \\
\mathbf{v_b} &= xy\hat{x} + 2yz\hat{y} + 3zx\hat{z} \\
\mathbf{v_c} &= y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}
\end{aligned}$$

Answer

$$
\begin{aligned}
\nabla \cdot \mathbf{v_a} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot (x^2\hat{x} + 3xz\hat{y} - 2xz\hat{z}) \\
&= 2x - 2x \\
&= 0.
\end{aligned}
$$

$$\nabla \cdot \mathbf{v_b} = y + 2z + 3x$$

$$\nabla \cdot \mathbf{v_c} = 2x + 2y$$

Question

4. Calculate the curls of the vector functions in problem 3.

Answer

$$
\begin{aligned}
&\underline{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \\
&\nabla \times \underline{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{x}\left(\partial/\partial y(-2xz) - \partial/\partial z\left(3xz^2\right)\right) \\
&- \hat{y}\left(\partial/\partial x(-2xz) - \partial/\partial z\left(x^2\right)\right) \\
&+ \hat{z}\left(\partial/\partial x\left(3xz^2\right) - \partial/\partial y\left(x^2\right)\right) \\
&= \hat{x}(-6xz) - \hat{y}(-2z) + \hat{z}\left(3z^2\right) \\
&= -6x\hat{x} + 2z\hat{y} + 3z^2\hat{z}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{v}_b = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z} \\
&\nabla \times \underline{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{x}(\partial/\partial y(3zx) - \partial/\partial z(2yz)) \\
&- \hat{y}(\partial/\partial x(3zx) - \partial/\partial z(xy)) \\
&+ \hat{z}(\partial/\partial x(zyz) - \partial/\partial y(xy)) \\
&= \hat{x}(-2y) - \hat{y}(3z) + \hat{z}(-x) \\
&= -2y\hat{x} - 3z\hat{y} - x\hat{z}
\end{aligned}
$$

$$\underline{v}_c = y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + 2yz\hat{z}$$

$$\begin{aligned}
\nabla \times \underline{v}_c = &\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy + z^2 & 2y^z \end{vmatrix} \\
=&\hat{x}\left(\partial/\partial y(2yz) - \partial/\partial z\left(2xy + z^2\right)\right) \\
&- \hat{y}\left(\partial/\partial x(2yz) - \partial/\partial z\left(y^2\right)\right) \\
&+ \hat{z}\left(\partial/\partial x\left(2xy + z^2\right) - \partial/\partial y\left(y^2\right)\right) \\
=&(2z - 2z)\hat{x} - 0\hat{y} + (2y - 2y)\hat{z} \\
=&\underline{0}
\end{aligned}$$

**Question**

5. Calculate the line integral of the function $\mathbf{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$ from the origin to the point (1,1,1) by three different routes:

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$
(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$
(c) the straight line.
(d) What is the line integral around the closed loop that goes out along path (a) and back along path (b)?

**Answer**

$$\underline{v} = x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}$$

$$\int_P \underline{v} \cdot d\underline{l}$$

$$d\underline{\ell} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

a) three sections of path to integrate: $(0,0,0) \to (1,0,0) \to ((,1,0) \to (1,1,1)$
Part i: $d\underline{l} = dx\hat{x} \quad y = 0, z = 0$

$$\begin{aligned}
\int_{\text{path}} x^2\hat{x} \cdot dx\hat{x} &= \int_0^1 x^2 dx \\
&= \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.
\end{aligned}$$

Part ii: $d\underline{l} = dy\hat{y} \quad x = 1, z = 0$

$$\int_{\text{path ii}} 2yz\hat{y} \cdot dy\hat{y} = \int_0^1 2yzdy$$
$$= 0$$

Part iii: $d\underline{l} = dz\hat{z} \quad x = 1, \quad y = 1$

$$\int_{\text{path } iii} y^2\hat{z} \cdot dz\hat{z} = \int_0^1 y^2 dz = \int_0^1 dz$$

$$\text{Total path } = \int_p \underline{v} \cdot d\underline{l} = 4/3$$

b) Path $(0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1)$

Path i $y = 0, x = 0 \quad d\underline{\ell} = dz\hat{z}$

$$\int_{\text{Path}} y^2\hat{z} \cdot dz\hat{z} = \int_0^1 0dz = 0$$

path_ii $z = 1, x = 0 \quad d\underline{\ell} = dy\hat{y}$

$$\int_{\text{path ii}} 2yz\hat{y} \cdot dy\hat{y} = \int_0^1 2ydy = y^2\Big|_0^1$$
$$= 1$$

Path iii: $z = 1, y = 1 \quad dl = dx\hat{x}$

$$\int_{\text{path iii}} x^2\hat{x} \cdot dx\hat{x} = \int_0^1 x^2 dx = \frac{x^3}{3}\Bigg|_0^1$$
$$= \frac{1}{3}$$

$$\text{Total path } = \int_p \underline{v} \cdot d\underline{l} = 4/3.$$

c)

$$x = y = z \quad , \quad dx = dy = dz$$
$$\int_p \underline{v} \cdot d\underline{\ell}$$
$$= \int (x^2\hat{x} + 2yz\hat{y} + y^2\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$
$$= \int x^2 dx + 2yzdy + y^2 dz$$

now write all in terms of one variable.

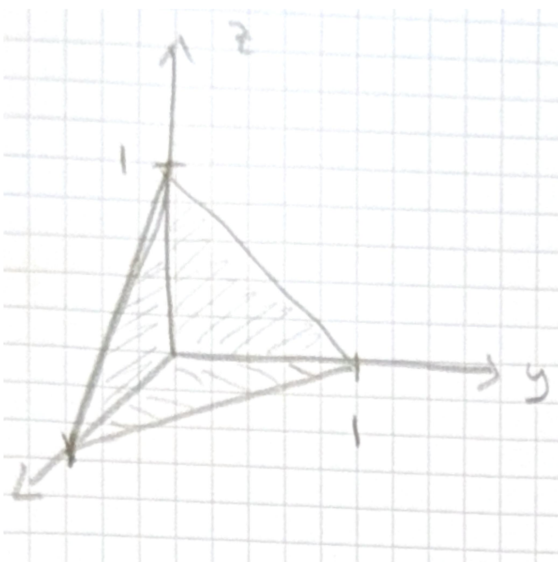
$$
\begin{aligned}
&= \int x^2dx + 2x^2dx + x^2dx \\
&= \int_0^1 4x^2dx \\
&= \frac{4}{3}x^3\Big|_0^1 = \frac{4}{3}
\end{aligned}
$$

d) $\oint v \cdot dl = 4/3 - 4/3 = 0$.

Question

6. Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

Answer




$$
\begin{aligned}
\int_V T d\tau &= \int_V z^2 dxdydz \\
&= \iiint z^2 dxdydz
\end{aligned}
$$

If we integrate $x$ first minimum is 0 max is at surface which is defined by $x + y + z = 1$. Thus, $x = 1 - y - z$. Once $x$ has been integrated $y$ also has minimum of $O$ and now max at line $y+z=1$. Finally we have just $z$ to consider which runs from 0 to 1 .

$$\begin{aligned}
&\int_0^1 \int_0^{1-z} \int_0^{1-z-y} z^2 dxdydz \\
&\int_0^1 \int_0^{1-z} z^2 x\Big|_{x=0}^{x=1-z-y} dydz \\
&= \int_0^1 z^2 \int_0^{1-z} 1-z-ydydz \\
&= \int_0^1 z^2 \left(y - zy - \frac{y^2}{2}\right)\Big|_0^{y=1-z} dz \\
&= \int_0^1 z^2 \left(1-z-z(1-z)-\frac{1}{2}(1-z)^2\right) dz
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 z^2 \left(1-z-2+z^2-\frac{1}{2}\left(1-2z+z^2\right)\right) dz \\
&= \int_0^1 z^2 \left(1-2z+z^2-\frac{1}{2}+z-\frac{z^2}{2}\right) dz \\
&= \int_0^1 z^2 \left(\frac{1}{2}-z+\frac{z^2}{2}\right) dz \\
&= \int_0^1 \frac{z^2}{2}-z^3+\frac{z^4}{2} dz \\
&= \frac{z^3}{6}-\frac{z^4}{4}+\frac{z^5}{10}\Big|_0^1 \\
&= \frac{1}{6}-\frac{1}{4}+\frac{1}{10}=\frac{1}{60}
\end{aligned}$$

## 15.2 Week 3

**Question**

1. Check the fundamental theorem of gradients with the scalar function $T=x^2+4xy+2yz^3$. Assume point a is at the origin and point $b=(1,1,1)$. Check the theorem for three paths:

a) $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (1,1,1)$

b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

c) the parabolic path $z = x^2$ and $y = x$

Answer

Part a) Path i) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$$d\underline{l} = dx\hat{x} \quad y = 0, z = 0$$

$$\int_{\text{path i}} \nabla T \cdot d\underline{l} = \int_0^1 2x dx = x^2\Big|_0^1 = 1$$

$$\text{Path ii) } d\underline{l} = dy\hat{y} \quad x = 1, z = 0$$

$$\int_{\text{path } ii} \nabla T \cdot d\underline{l} = \int_0^1 4dy = 4y\Big|_0^1 = 4.$$

Path iii) $d\underline{\ell} = dz\hat{z} \quad x = 1, y = 1$

$$\int_{\text{path iii}} \nabla T \cdot d\underline{l} = \int_0^1 6z^2 dz = 2z^3\Big|_0^1 = 2$$

Total path $1 + 4 + 2 = 7$

Part b)

$$(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$$

$$\text{Path i} \quad d\underline{l} = dz\hat{z} \quad y = 0, x = 0$$

$$\int \nabla T \cdot dl = \int_0^1 0dz = 0$$

$$\text{Path ii) } d\underline{l} = dy\hat{y} \quad x = 0, z = 1$$

$$\int \nabla T \cdot d\underline{l} = \int_0^1 2dy = 2y\Big|_0^1 = 2$$

Path iii) $\quad d\underline{l} = dx\hat{x} \quad y = 1, z = 1$

$$\int \nabla T \cdot d\underline{l} = \int_0^T (2x + 4)dx = x^2 + 4x\Big|_0^1 = 5$$

Total path 7.

Part c) parabolic path $z = x^2, y = x \quad dx = dy \quad dz = 2xdx$

$$d\underline{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\begin{aligned}\nabla T \cdot d\underline{l} =& (2x+4y)dx + \left(4x+2z^3\right)dy + 6yz^2 dz \\ &= (2x+4x)dx + \left(4x+2x^6\right)dx + 6xx^4 \cdot 2xdx \\ &= (6x+4x+2x^6+12x^6)dx.\end{aligned}$$

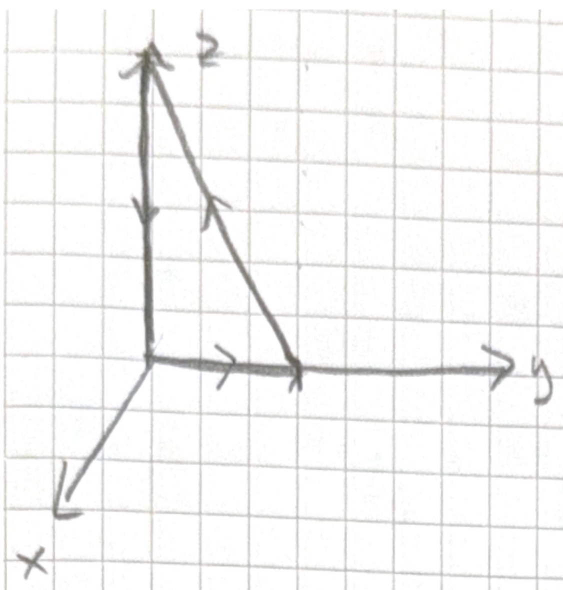
$$\int \nabla T \cdot dl = \int_0^1 \left(10x + 14x^6\right) dx$$

$$5x^2 + 2x^7\Big|_0^1 = 7$$

**Question**

2. Compute the line integral of $\mathbf{v} = 6\,\hat{x} + yz^2\,\hat{y} + (3y+z)\,\hat{z}$ along the triangular path $(0,0,0) \to (0,1,0) \to (0,0,2) \to (0,0,0)$. Check your answer using Stokes theorem.

**Answer**




$$\mathbf{v} = 6\hat{x} + y^2\hat{y} + (3y+z)\hat{z}$$

$$(0,0,0) \to (0,1,0) \to (0,0,2)$$
$$d\underline{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

Path i $x = 0, z = 0 \quad d\underline{l} = dy\hat{y}$

$$\int \underline{v} \cdot d\underline{l} = \int yz^2 dy \int_{\text{path i}} \mathbf{v} \cdot d\underline{l} = \int_0^1 0 = 0$$

path ii $x = 0 \quad d\underline{l} = dy\hat{y} + dz\hat{z}$

$$z = my + b$$
$$z = -2y + 2 \quad dz = -2dy$$
$$\int_{\text{path ii}} v \cdot d\underline{l} = \int yz^2 dy + \int 3y + zdz$$
$$= \int y(-2y+2)^2 dy + \int (3y + -2y + 2) \cdot -2dy$$
$$= \int_1^0 4y^3 - 8y^2 + 4y - 2y - 4dy$$
$$= \int_1^0 4y^3 - 8y^2 + 2y - 4dy$$
$$= -\frac{2}{3}\, y^4 - \frac{8}{3} y^3 + y^2 - 4y \Big|_1^0$$
$$= \frac{2}{3} + 4 = \frac{14}{3}$$

part iii) $d\underline{l} = dz\hat{z} \quad y = 0, x = 0$

$$\underline{v} \cdot d\underline{l} = 3y + zdz$$
$$= zdz$$
$$\int_2^0 zdz = \frac{z^2}{2}\Big|_2^0$$
$$= -2$$

Total path

$$\oint \mathbf{v} \cdot d\underline{l} = \frac{14}{3} - \frac{6}{3} = \frac{8}{3}$$

Check with Stokes Theorem:

$$\int_s (\nabla \times \underline{v}) \cdot d\underline{a}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix}$$

$$= \hat{x}(3-2yz) - \hat{y}(0-0) + \hat{z}(0-0)$$

$$= (3-2y^2)\,\hat{x}$$

$$\iint (3-2yz)dydz \quad z = -2y+2$$

$$\int_0^1 \left[\int_0^{-2y+2} 3 - 2yzdz\right] dy$$

$$\int_0^1 \left[3z - yz^2\Big|_{z=0}^{z=2-2y}\right] dy$$

$$\int_0^1 6 - 6y - y(2-2y)^2 dy$$

$$6y - 5y^2 + \frac{8}{3}y^3 - y^4\Big|_0^1$$

$$6 - 6y - y\left(4 - 8y + 4y^2\right) dy$$

$$\int_0^1 6 - 6y - 4y + 8y^2 - 4y^3 dy$$

$$6 - 5 + \frac{8}{3} - 1$$

$$= \frac{8}{3}$$

Question

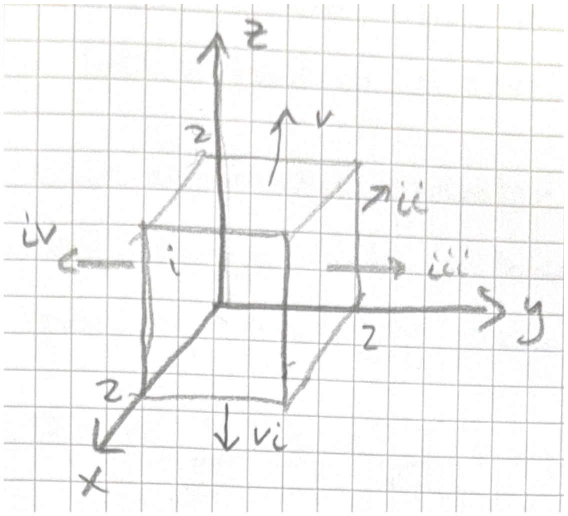
3.

a) If **A** and **B** are vector functions what does $(\mathbf{A} \cdot \nabla)\mathbf{B}$ mean? (What are the x, y, and z components?)
b) Compute $(\hat{r} \cdot \nabla)\hat{r}$.
c) For the functions from problem 3 in week 2 evaluate $(\mathbf{v_a} \cdot \nabla)\mathbf{v_b}$

Question

4. Test the divergence theorem for the function $\mathbf{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$. Take for the volume a cube placed ath the origin with sides of length 2.

Answer




$\underline{v} = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$

$$\int_V (\nabla \cdot \underline{v}) d\tau = \oint_s \underline{v} \cdot d\underline{a}$$
$$\nabla \cdot \underline{v} = y + 2z + 3x$$
$$d\tau = dxdydz$$
$$\int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) dxdydz$$
$$\int_0^2 \int_0^2 yx + 2zx + \frac{3x^2}{2}\Big|_{x=0}^{x=2} dydz$$
$$\iint_0^2 (2y + 4z + 6) dydz$$
$$\int_0^2 y^2 + 4zy + 6y|_{y=0}^{y=z} dz$$
$$\int_0^2 4 + 8z + 12dz$$
$$16z + 4z^2|_0^2$$
$$3z + 16 = 48$$

i) $d\underline{a} = dydz\hat{x}$

$$\underline{v} \cdot \underline{a} = xydydz$$

$$\int_0^2 \int_0^2 xydydz$$

$$= \int_0^2 \frac{xy^2}{2}\bigg|_{y=0}^{y=2} dz$$

$$\int_0^2 2xdz$$

$$= 2xz|_0^2 \quad x = 2$$

$$= 8.$$

(ii) $d\underline{a} = -dydz\hat{x} \quad x = 0$

$$\int_0^2 \int_0^2 xydydz = 0.$$

iii) $d\underline{a} = dxdz\hat{y} \quad y = 2.$

$$\underline{v} \cdot d\underline{a} = 2 \cdot 2zdxdz$$

$$\int_0^2 \int_0^2 4zdxdz$$

$$\int_0^2 4zx\bigg|_{x=0}^{x=2} dz$$

$$\int_0^2 8zdz$$

$$4z^2|_0^2 = 16$$

(iv) $d\underline{a} = -dxdz\hat{y} \quad y = 0$

$$\int_0^2 \int_0^2 0dxdz = 0$$

v) $d\underline{a} = dxdy\hat{z} \quad z = 2$

$$\int_0^2 \int_0^2 6xdxdy$$

$$\int_0^2 3x^2\bigg|_{x=0}^{x=2} dy$$

$$= 12y|_0^2 = 24.$$

$$\text{vi) } d\underline{a} = -dxdy\hat{z}z = 0$$

$$\int_\delta 0dxdy = 0$$

$$\therefore \oint_S \underline{v} \cdot d\underline{a} = 48\cdot$$

**Question**

5. Test Stokes theorem for the **v** in problem 4. Take the path to be three segments that make a triangular loop: $(0,0,0) \to (0,2,0) \to (0,0,2) \to (0,0,0)$.

**Answer**

$$\int_S (\nabla \times \underline{v}) \cdot d\underline{a} = \oint_P \underline{v} \cdot d\underline{l}$$

Right hand side first:

$$v = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$$

$$\nabla \times \underline{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{x}(0 - 2y) - \hat{y}(3z - 0) + \hat{z}(0 - x)$$

$$= -2y\hat{x} - 3z\hat{y} - x\hat{z}$$

$$my + b = z$$

$$d\underline{a} = dydz\hat{x}$$

$$z = -y + 2$$

$$\iint (-2y\hat{x} - 3z\hat{y} - x\hat{z}) \cdot dydz\hat{x}$$

$$= \int_{y=0}^{y=2} \int_{z=0}^{z=2-y} -2ydzdy$$

$$\int_0^2 - 2yz\Big|_{z=0}^{z=2-y} dy$$

$$\int_0^2 -2y(2 - y)dy$$

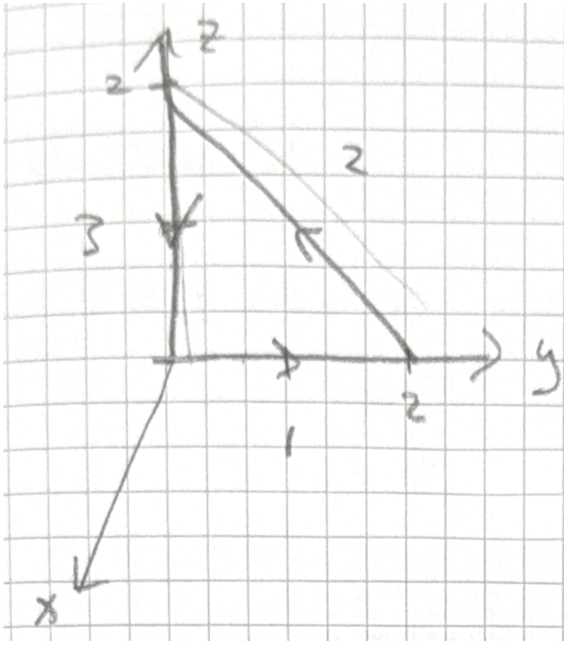
$$- \int_0^2 4y - 2y^2 dy$$

$$- \left( 2y^2 - \frac{2y^3}{3}\Bigg|_0^2 \right)$$

$$- \left( 8 - \frac{16}{3} \right)$$

$$- \frac{24 - 16}{3} = -\frac{8}{3}$$

Now for the path side ...




Section 1 of the path:

$$d\underline{l} = dy\hat{y} \quad x = 0, z = 0$$
$$\underline{v} \cdot d\underline{l} = 2yz = 0$$
$$\int_0^2 2yzdy = \int_0^2 0 = 0$$

Part 2 of path:

$$x = 0, \quad d\underline{l} = dy\hat{y} + dz\hat{z} \quad z = -y + 2 \quad dz = -dy$$
$$\int 2yzdy + \int^{3xzdz}$$
$$\int_2^0 2y(-y+2)dy$$
$$\int_2^0 -2y^2 + 4ydy$$
$$-\frac{2}{3}y^3 + 2y^2\Big|_2^0$$
$$\frac{2}{3} \cdot 8 - 8$$
$$\frac{16}{3} - \frac{24}{3} = -\frac{8}{3}$$

Part 3 of path:

$$d\underline{l} = dz\hat{z} \quad x = 0, y = 0$$

$$\int \underline{v} \cdot d\underline{l} = \int_2^0 3zxdz = 0$$

So total path

$$\oint_p \underline{v} \cdot d\underline{l} = -\frac{8}{3}.$$

**Question**

6. Prove that the divergence of the curl is always zero. Check it for the function $\mathbf{v_a}$ of problem 3 in week 2.

**Answer**

$$\nabla \cdot (\nabla \times \mathbf{v})$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\partial v_z/\partial y - \partial v_y/\partial z\right)$$

$$- \hat{y}\left(\partial v_z/\partial x - \partial v_x/\partial z\right)$$

$$+ \hat{z}\left(\partial v_y/\partial x - \partial v_x/\partial y\right)$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_y}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \mathbf{v})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_x}{\partial y}\right)$$

$$\frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial z \partial y}$$

$$= 0.$$

**Question**

7. Prove that the curl of the gradient is always zero.

Answer

$$
\begin{aligned}
&\nabla \times \nabla T \\
&\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \\
&\nabla \times \nabla T = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \\
&= \hat{x}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial y \partial z}\right) - \hat{y}\left(\frac{\partial^2 T}{\partial x \partial z} - \frac{\partial^2 T}{\partial x \partial z}\right) \\
&+ \hat{z}\left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial x \partial y}\right) = 0
\end{aligned}
$$

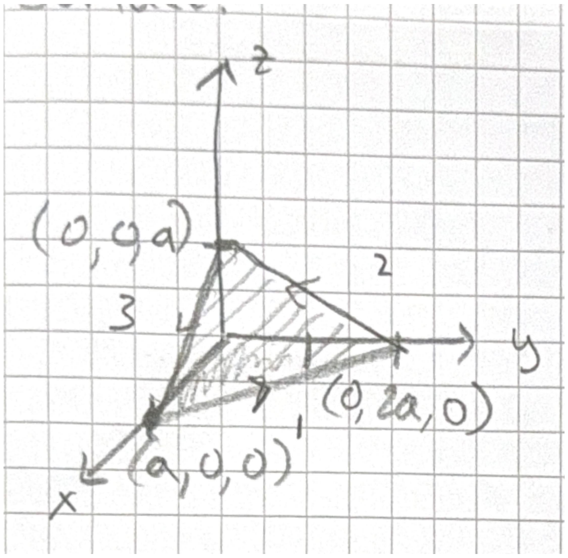
Question

8. Use Stokes theorem to find the surface integral $\mathbf{v} = y\,\hat{z}$ for the triangular surface defined by the vertices $(a, 0, 0), (0, 2a, 0), (0, 0, a)$. Hint one side of Stokes theorem will be much more straight forward than the other.

Answer

$$
\begin{gathered}
\underline{v} = y\hat{z} \\
\int_S (\nabla \times \underline{v}) \cdot d\underline{a} = \oint_p \underline{v} \cdot d\underline{l}
\end{gathered}
$$

Surface:




$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \partial\partial y & \partial\partial z \\ 0 & 0 & y \end{vmatrix}$$

$$= \hat{x}(1) - \hat{y}0 + \hat{z}0$$

$$\nabla \times \underline{v} = \hat{x}$$

Part 1: $d\underline{l} = dx\hat{x} + dy\hat{y} \quad z = 0 \quad dz = 0$
$\int y\hat{z} \cdot (dx\hat{x} + dy\hat{y}) = 0$

Part 2: $d\underline{l} = dy\hat{y} + dz\hat{z} \quad x = 0$

$$z = my + b$$
$$z = -1/2y + b$$
$$z = -1/2y + 1$$
$$dz = -1/2dy$$

$$\int y\hat{z} \cdot (dy\hat{y} + dz\hat{z})$$

$$\int ydz = \int_{2a}^{0} y \cdot -1/2dy$$

$$= -\frac{1}{2}\frac{y^2}{2}\bigg|_{2a}^{0}$$

$$= \frac{1}{4}4a^2 = a^2$$

Part 3: $d\underline{l} = dx\hat{x} + d\hat{z} \quad y = 0,$

$$\int y\hat{z} \cdot (dx\hat{x} + dz\hat{z}) = 0$$

## 15.3 Week 4

**Question**

1. $r = e^{-p^2-q^2}$, $p = e^s$, $q = e^{-s}$. Find $\frac{dr}{ds}$.

**Answer**

There are several ways to do this - here is one way:

$$\frac{dr}{ds} = \frac{\partial r}{\partial p}\frac{dp}{ds} + \frac{\partial r}{\partial q}\frac{dq}{ds}$$
$$\frac{\partial r}{\partial p} = e^{-p^2-q^2} \cdot -2p$$
$$\frac{\partial r}{\partial q} = e^{-p^2-q^2} \cdot -2q$$
$$\frac{dp}{ds} = e^s$$
$$\frac{dq}{ds} = -e^{-s}$$

Substitute all of this into the full derivative

$$\begin{aligned}\frac{dr}{ds} &= -2pe^{-p^2-q^2} \cdot e^s + (-2q)e^{-p^2-q^2} \cdot -e^{-s} \\ &= (-2pe^s + 2qe^{-s})e^{-p^2-q^2} \\ &= 2r(q^2 - p^2)\end{aligned}$$

**Question**

2. $c = \sin(a - b)$, $b = ae^{2a}$. Find $\frac{dc}{da}$.

Answer

$$
\begin{aligned}
\frac{dc}{da} &= \frac{\partial c}{\partial b} \cdot \frac{db}{da} + \frac{\partial c}{\partial b} \cdot \frac{da}{da} \\
\frac{\partial c}{\partial b} &= \cos(a-b) \cdot -1 \\
\frac{\partial c}{\partial a} &= \cos(a-b) \\
\frac{db}{da} &= e^{2a} + ae^{2a} \cdot 2 = e^{2a}(1+2a) \\
\therefore \frac{dc}{da} &= \cos(a-b)(1-2b-e^{2a})
\end{aligned}
$$

Question

3. $z = x^2 + 2y^2$

   a. Find $\left(\frac{\partial z}{\partial x}\right)_y$.
   b. Write $x$ and $y$ in plane polar coordinates (in terms of $r$ and $\theta$) and find $\left(\frac{\partial z}{\partial x}\right)_r$.

Answer

a. Take the partial derivative of $z$ keeping $y$ constant: $\left(\frac{\partial z}{\partial x}\right)_y = 2x$.
b. There are several approaches to this problem - here is mine. First write $x$ and $y$ in plane polar coordinates:

$$x = r\cos\theta \quad y = r\sin\theta$$

   Since we want the partial of $z$ with respect to $x$ while keeping $r$ constant lets take the opportunity to substitute for $y$ in terms of $\theta$ and $r$.

$$
\begin{aligned}
z &= x^2 + 2y^2 \\
z &= x^2 + 2r^2\sin^2\theta
\end{aligned}
$$

We can substitute again to get rid of the $\theta$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos\theta = \frac{x}{r} \qquad \cos^2\theta = \frac{x^2}{r^2}$$

$$\sin^2\theta = 1 - \frac{x^2}{r^2}$$

$$z = x^2 + 2r^2\left(1 - \frac{x^2}{r^2}\right)$$

$$= -x^2 - 2r^2$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)_r = -2x$$

Question

4. Solve

$$\int_{x=0}^{1}\int_{y=2}^{4} 3x\,dxdy$$

Answer

$$\int_{x=0}^{1} 3x\,dx \int_{y=2}^{4} dy$$

$$\frac{3}{2}x^2\Big|_0^1 \cdot y\Big|_2^4 = 3$$

Question

5. Solve

$$\int_{y=0}^{2}\int_{x=2y}^{4} dxdy$$

Answer

$$\int_{y=0}^{2} (4-2y)dy$$

$$4y - y^2\Big|_0^2 = 4$$

Question

6. Solve

$$\int_{x=0}^{1}\int_{y=x}^{e^x} y\,dxdy$$

Answer

$$=\int_{x=0}^{1}\frac{y^2}{2}\bigg|_{x}^{e^x}\,dx$$

$$=\frac{1}{2}\int_{0}^{1}(e^{2x}-x^2)dx$$

$$=\frac{1}{2}\left(\frac{e^{2x}}{2}-\frac{x^3}{3}\right)\bigg|_{0}^{1}=\frac{e^2}{4}-\frac{5}{12}$$

Question

7. $\iint\limits_A (2x-3y)\,dxdy$, where $A$ is the triangle with vertices $(0,0),(2,1),(2,0)$.

Answer

The hypotenuse of the triangle is the line $y=\frac{1}{2}x$. Lets integrate over $y$ first keeping $x$ constant.

$$\int_{x=0}^{2}\int_{y=0}^{\frac{x}{2}}(2x-3y)\,dydx$$

$$\int_{x=0}^{2}(2xy-\frac{3}{2}y^2)\bigg|_{y=0}^{\frac{x}{2}}\,dx$$

$$\int_{0}^{2}\frac{5}{8}x^2\,dx=\frac{5}{3}$$

Question

8. Solve

$$\iint 2xy\,dxdy$$

over the triangle $(0,0),(2,1),(3,0)$.

Answer

I split this into two right triangles. One with $x$ ranging from 0 to 2 with area $A_1$ and a hypotenuse $y = x/2$ and one with $x$ ranging from 2 to 3, area $A_2$ and a hypotenuse $y = 3 - x$.

$$
\begin{aligned}
&= \iint_{A_1} 2xy\,dxdy + \iint_{A_2} 2xy\,dxdy \\
&= \int_{x=0}^{2}\int_{y=0}^{\frac{x}{2}} 2xy\,dydx + \int_{x=2}^{3}\int_{y=0}^{3-x} 2xy\,dydx \\
&= \int_0^2 xy^2\Big|_{y=0}^{\frac{x}{2}} dx + \int_2^3 xy^2\Big|_{y=0}^{3-x} dx \\
&= \int_0^2 x\cdot\frac{x^2}{4}dx + \int_0^3 x(3-x)^2dx \\
&= \int_0^2 \frac{x^3}{4}dx + \int_2^3 x(9-6x+x^2)dx \\
&= \frac{x^4}{16}\Big|_0^2 + \int_2^3 (9x-6x^2+x^3)dx \\
&= 1 + \left(\frac{9x^2}{3} - \frac{6x^3}{3} + \frac{x^4}{4}\right)\Big|_2^3 = \frac{7}{4}
\end{aligned}
$$

Question

9. Solve

$$\int_{x=1}^{2}\int_{y=x}^{2x}\int_{z=0}^{y-x} dzdydx$$

Answer

$$
\begin{aligned}
&= \int_{x=1}^{2} \int_{y=x}^{2x} (y-x)\,dydx \\
&= \int_{1}^{2} \left( \frac{y^2}{2} - xy \right) \Bigg|_{x}^{2x} \\
&= \int_{1}^{2} \frac{x^2}{2}\,dx = \frac{7}{6}
\end{aligned}
$$

## 15.4 Week 5 (due 12:30pm Friday 18 Oct)

1. Calculate the Laplacian of $\ln(x^2+y^2)$.
2. Is $\mathbf{F} = y\hat{i} + xz\hat{j} + z\hat{k}$ a conservative field? Justify your answer.

   Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ from $(0,0,0)$ to $(1,1,1)$ along two paths:

   a. a broken line $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$
   b. a straight line
3. Given $\phi = z^3 - 3xy$

   a. find grad $\phi$
   b. find the directional derivative of $\phi$ at the point(1, 2,3) in the direction $\hat{i} + \hat{j} + \hat{k}$
4. Let $\mathbf{v} = x\hat{i} + y\hat{j} + z\hat{k}$. Evaluate $\oint_S \mathbf{v} \cdot d\mathbf{a}$ over the closed surface of a cylinder of hight $h$ and radius $a$ with a base centred at the origin. (Hint: consider using the divergence theorem.)
5. Calculate the volume of the region bounded by the planes $z = 2x + 3y + 6$ and $z = 2x + 7y + 8$ and the triangle with vertices $(0,0)$, $(0,3)$, and $(2,1)$ projected in the vertical direction (taken as the z-direction).