Mathematical Physics PHYS23020

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Preface

This text covers the lectures for Part A of Mathematical Physics PHYS23020. If you notice any errors please email me zoe.leinhardt@bristol.ac.uk.

Part I

Course Notes

1 Introduction

Welcome to second year Mathematical Physics. This unit is divided into two sections Part A - Differentiation, Integration and Vector Calculus and Part B - Linear Algebra and Fourier Series.

1.1 Course logistics

You should familiarize yourself with the blackboard page for this unit. This is the main source of information for this unit. The lecture and problems/example class schedule should appear in your personal timetable.

Lectures will be delivered in person three times a week along with one problems class a week starting in week 2.

1.2 Textbooks

There are several textbooks that you may find useful: Mathematical methods in the physical sciences by Mary Boas and Mathematical methods for physics and engineering by Riley, Hobson, and Bence. Once we move on to vector calculus you may also like to work through some of the first chapter of Introduction to Electrodynamics by Griffiths. All of these texts are listed in the reading list and are available digitally and physically in the University Library.

2 Partial Differentiation

You have seen functions of a single variable often - consider $f(x) = ax + 2$ for example. This function contains a constant 2 and a parameter a but only x is a variable and derivatives of $f^{(n)}(x) = \frac{d^n f}{dx^n}$. But we can also have functions that depend on more than one variable like $f(x, y) = x^2 + 3xy$ which depends on two variables x and y. In this case for any pair of values x and y $f(x, y)$ has a well defined value. For example, $f(2, 3) = 4 + 3 * 2 * 3 = 22$.

We can extend this idea to functions of more than two variables. For the n-variable case $f(x_1, x_2, ...x_n)$ for a function that depends on $x_1, x_2, ...x_n$. Functions of one variable can be represented by a graph on a sheet of paper. Functions of two variables can be represented by a surface in 3D space. For example, you may picture $f(x, y)$ as describing height and position in a landscape. Functions of many variables however are very hard to visualise often. To make things easier in this regard we will focus on functions of two variables most of the time.

2.1 Definition of the partial derivative

A function of two variables $f(x, y)$ will have a gradient in all directions in the xy plane. Lets consider finding the rate of change of $f(x, y)$ in the positive x- and y-directions. These rates of change are called partial derivotives with respect to x and y respectively and have lots of important applications.

For $f(x, y)$ we define the derivative with respect to x by saying that it is that for a one variable function when y is held fixed and treated as constant. We write this as $\frac{\partial f}{\partial x}$ and read it as the partial derivative of f with respect to x. We can do the same for $y \to \frac{\partial f}{\partial y}$.

To define the partial derivative formally

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

provided the limit exists. Similarly,

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

It is common when discussing partial derivatives involving functions of multiple variables to indicate what variable is being held fixed

$$\left(\frac{\partial f}{\partial x}\right)_y \text{ and } \left(\frac{\partial f}{\partial y}\right)_x$$

You can extend this to functions of multi variables

$$\frac{\partial f\left(x_1 \dots x_n\right)}{\partial x_i}.$$

Just as for single variable functions second and higher order partial derivatives are possible:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

Note these last two are equal provided that the second partial derivatives are continuous at the point in question.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\
\frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{\partial^2 f}{\partial x_j \partial x_i}
\end{aligned}$$

> **Question**
>
> Find the first and second derivatives of $f(x, y) = 2x^3y^2 + y^3$.

Answer

$$\frac{\partial f}{\partial x} = 6x^2y^2$$
$$\frac{\partial f}{\partial y} = 4x^3y + 3y^2$$
$$\frac{\partial^2 f}{\partial x^2} = 12xy^2$$
$$\frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y$$
$$\frac{\partial^2 f}{\partial x \partial y} = 12x^2y$$
$$\frac{\partial^2 f}{\partial y dx} = 12x^2y$$

2.2 The total differential and total derivative

In the previous section we defined the rate of change of f along the positive $x-$ and $y-$ axes. Now lets consider the rate of change of $f(x,y)$ in an arbitrary direction. Lets make simultaneous small changes in Δx in x and Δy in y so that f changes to $f + \Delta f$. Then we have

$$\begin{aligned}
\Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y) \\
&= \left[\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right]\Delta x \\
&\quad + \left[\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right]\Delta y \\
\Delta f &\approx \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y
\end{aligned}$$

Letting the small changes Δx and $\Delta y \Rightarrow$ very very small, we define the total derivative as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

extended to n variables $f\left(x_1, x_2, \dots x_n\right)$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \dots \frac{\partial f}{\partial x_n}dx_n$$

Question

Find the total differential of the function

$$f(x, y) = y\exp(x + y)$$

Answer

$\frac{\partial f}{\partial x} = y\exp(x + y)$, $\frac{\partial f}{\partial y} = \exp(x + y) + y\exp(x + y)$

$$\begin{aligned} df &= [y\exp(x + y)]dx + [\exp(x + y) + y(\exp(x + y))]dy \\ &= [y\exp(x + y)]dx + [(1 + y)\exp(x + y)]dy \end{aligned}$$

Sometimes even if there are multiple variables they can be represented in terms of one variable. For example,

$$x_i = x_i\left(x_1\right) \quad 2, 2, 3, \ldots n$$

This means that f can then be expressed as a function of x_1. The total derivative of f can then be obtained by ordinary differentiation. Alternatively,

$$\left(\frac{df}{dx}\right) = \left(\frac{\partial f}{\partial x_1}\right) + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dx_1} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dx_1}$$

The partial derivative, $\frac{\partial f}{\partial x_1}$, only accounts for explicit appearances of x_1 and no allowance must be made for the knowledge that changing x_1 necessarily changes $x_2, x_1 \ldots x_n$. The contributions of these are the other component of the RHS.

Question

Find the total derivative of $f(x, y) = x^2 + 3xy$ with respect to x given that $y = \sin^{-1} x$.

2.3 Exact and inexact differentials

What if we want to find a function f that differentiates to give a known differential?

Often this relies on experience. For example, consider $df = x\,dy + y\,dx$ is $f(x, y) = xy + c$ where c is a constant.

Differentials like this (those that integrate directly) are called exact differentials where as those that do not are called inexact differentials. Consider

$$df = xdy + 3ydx$$

is not an obvious differential of any function. Note it is possible to make an inexact differential exact by multiplying it by an integrating factor (not going to deal that in this course).

Question

Show that $df = x\,dy + 3y\,dx$ is inexact.

Answer

If we integrate with respect to x

$$f(x, y) = 3xy + g(y)$$

where $g(y)$ is any function of y. If we integrate with respect to y

$$f(x, y) = xy + h(x)$$

where $h(x)$ is any function of x. These conclusions are inconsistent with each other thus the differential is inexact.

What makes a differential exact?

$$df = A(x, y)dx + B(x, y)dy$$
$$\frac{\partial f}{\partial x} = A(x, y), \frac{\partial f}{\partial y} - B(x, y)$$

using the property

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$
$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

or

$$\frac{\partial}{\partial x}B(x, y) = \frac{\partial}{\partial y}A(x, y)$$

This is both necessary and sufficient for a differential to be exact.

So now lets return to the previous example and lets show that $xdy + 3ydx$ is inexact

$$B(x, y)dy + A(x, y)dx$$

$$\frac{\partial A}{\partial y} = 3, \frac{\partial B}{\partial x} = 1$$
$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \text{ thus this differential is inexact.}$$

This can be extended to many variables by requiring

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \text{ for all pairs } i, j$$

$$df = \sum_{i=1}^{n} g_i \left(x_1, x_i, ... x_n\right) dx_i$$

Question

Is $df = (y+z)dx + xdy + xdz$ an exact or inexact differential?

Answer

To answer this we need to check $1/2n(n-1)$ relationships to check or $\frac{1}{2} \cdot 3 \cdot 2 = 3$, relationships to check.

$$g_1 = y+z, g_2 = x, g_3 = x$$

$$\frac{\partial x}{\partial y} \stackrel{?}{=} \frac{\partial y}{\partial x}$$

$$\frac{\partial g_1}{\partial y} = 1 \quad \frac{\partial g_2}{\partial x} = 1$$

$$\frac{\partial x}{\partial z} \stackrel{?}{=} \frac{\partial z}{\partial x}$$

$$\frac{\partial g_1}{\partial z} = 1 \quad \frac{\partial g_3}{\partial x} = 1$$

$$\frac{\partial y}{\partial z} \stackrel{?}{=} \frac{\partial z}{\partial y}$$

$$\frac{\partial g_2}{\partial z} = 0 \quad \frac{\partial g_2}{\partial z} = 0.$$

All relationships are consistent so this differential is exact.

2.4 Chain Rule

Lets now consider the case where x and y are functions of another variable say u.

Lets say we want to find $\frac{df}{du}$. We could substitute $x(u)$ and $y(u)$ into $f(x, y)$ and then differentiate with respect to u but sometimes it is easier to use total differentials.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

we can divide by du $\frac{df}{du} = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du}$

This is called chain rule for partial differentiation. The expression above provides the total derivative of f with respect to u and is particularly helpful when an equation is expressed in parametric form.

Question

Given $x(u) = 1 + au$ and $y(u) = bu^3$ find the rate of change of $f(x, y) = xe^{-y}$ with respect to u.

Answer

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{-y} \\
\frac{\partial f}{\partial y} &= xe^{-y} \cdot -1 \\
\frac{dx}{du} &= a \\
\frac{du}{du} &= 3bu^2 \\
\frac{df}{du} &= e^{-y} \cdot a - xe^{-y} \cdot 3bu^2 \\
&= ae^{-bu^3} - (1 + au)e^{-bu^3} \cdot 3bu^2 \\
\frac{df}{du} &= e^{-bu^3}\left(a - 3bu^2 - 3abu^3\right)
\end{aligned}$$

For n variables each which are a function of u

$$\frac{df}{du} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{du} = \frac{\partial f}{\partial x_1}\frac{dx_1}{du} + \frac{\partial f}{\partial x_2}\frac{dx_2}{du} + \dots + \frac{\partial f}{\partial x_n}\frac{dx_n}{du}$$

2.5 Change of variables

It is sometimes useful to change variables during an analysis - can use chain rule to do this

$$\frac{\partial f}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial u_j}, j = 1, 2, \dots n$$

2.6 Thermodynamic relations

First Law of thermodynamics $\Rightarrow$ Conservation of energy

$$dU = TdS - PdV$$

U is internal energy, T is temperature, S is entropy, P is pressure and V is volume.

The four quantities on the right hand side are not independent – any two can be varied independently but then the other two are determined. We will for the moment look at this relation mathematically:

$$dU = \left(\frac{\partial U}{\partial X}\right)_Y dX + \left(\frac{\partial U}{\partial Y}\right)_X dY$$
$$\frac{\partial^2 U}{\partial X \partial Y} = \frac{\partial^2 U}{\partial Y \partial X}$$

where X and Y are chosen from P, V, S, and T.

Question

Show that $\left(\frac{\partial T}{\partial V}\right)_s = -\left(\frac{\partial P}{\partial S}\right)_V$

Answer

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$
$$TdS - PdV = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$
$$T = \left(\frac{\partial U}{\partial S}\right)_V \text{ and } -P = \left(\frac{\partial U}{\partial V}\right)_S$$
$$\frac{\partial^2 U}{\partial V \partial S} = \frac{\partial^2 U}{\partial S \partial V}$$

So if we now take the partial derivative T with respect to V and the partial of P with respect to S we find

$$\frac{\partial T}{\partial V} = \frac{\partial^2 U}{\partial V \partial S}$$
$$-\frac{\partial P}{\partial S} = \frac{\partial^2 U}{\partial S \partial V}$$
$$\therefore \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$$

Question

Show that $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

2.7 Differentiation of integrals

$$F(x,t) = \int f(x,t)\,dt$$
$$\frac{\partial F(x,t)}{\partial t} = f(x,t)$$

Assuming that the second derivatives are continuous

$$\frac{\partial^2 F(x,t)}{\partial t \partial x} = \frac{\partial^2 F(x,t)}{\partial x \partial t}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial F(x,t)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F(x,t)}{\partial t}\right) = \frac{\partial}{\partial t} f(x,t)$$

Integrating this with respect to t gives us

$$\frac{\partial F(x,t)}{\partial x} = \int \frac{\partial}{\partial x} f(x,t) dt$$

Now consider the definite integral

$$I(x) = \int_{t=u}^{t=v} f(x,s) dt$$
$$= F(x,v) - F(x,u)$$

where u and v are constants. Differentiating this integral with respect to x

$$\frac{\partial I(x)}{\partial x} = \frac{\partial F(x,v)}{\partial x} - \frac{\partial F(x,u)}{\partial x}$$
$$= \int^{v} \frac{\partial}{\partial x} f(x,t) dt - \int^{u} \frac{\partial}{\partial x} f(x,t) dt$$
$$\frac{\partial I(x)}{\partial x} = \int_{u}^{v} \frac{\partial}{\partial x} f(x,t) dt$$

This is Leibnitz' rule for differentiating integrals. It means that for constant limits of integration the order of differentiation and integration can be reversed. In the general case

$$
\begin{aligned}
I(x) &= \int_{t=u(x)}^{t+v(x)} f(x,t)dt \\
&= F(x,v(x)) - F(x,u(x)) \\
\frac{\partial I}{\partial v} &= f(x,v(x)) \text{ and } \frac{\partial I}{\partial u} = -f(x,u(x)) \\
\frac{dI}{dx} &= \left(\frac{\partial I}{\partial v}\right)\frac{dv}{dx} + \left(\frac{\partial I}{\partial u}\right)\frac{du}{dx} + \frac{\partial I}{\partial x} \\
&= f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \frac{\partial}{\partial x}\int_{u(x)}^{v(x)} f(x,t)dt \\
&= f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,t)dt
\end{aligned}
$$

Can do the final step because $u(x)$ and $v(x)$ are held constant in the last term.

Question

Find the derivative of

$$I(x) = \int_x^{x^2} \frac{\sin xt}{t} dt$$

3 Multiple Integrals

3.1 Multiple Integrals

For functions of multiple variables we may also consider integrals of multiple variables.

3.2 Integration (Review)

$I = \int_a^b f(x)dx$ can be thought of as the area under the function $f(x)$ from $f(a)$ to $f(b)$.

In this case I is a definite integral of $f(x)$ between the lower limit $x = a$ and upper limit $x = b$, and $f(x)$ is the integrand.

The definition of an integral as the area under the curve is not a formal definition - but it can be visualised, The formal definition of I involves subdividing the finite interval $a \leq x \leq b$ into a large number of subintervals $a = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_n = b$ and then forming a sum $\xi = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1})$ where x_i is an arbitrary point that lies in the range $\xi_{i-1} \leq x_i \leq \xi_i$. If n is now allowed to tend to infinity and the interval ξ_{i-1} to ξ_i tends to zero then S may or may not tend to a unique limit I. If it does then the definite integral $f(x)$ between a and b is defined as having the value I. If no unique limit exists the integral is undefined. For continuous functions and finite intervals $a \leq x \leq b$ the existence of a unique livest is assumed and the integral exists.

Question

Evaluate from first principles the integral

$$I = \int_0^b x^2 dx$$

Answer

Lets first approximate the area under the curve x^2 as a series of rectangles with width h. If we the the lower end of each subinterval to give the height then the area of the k^{th}

rectangle will be

$$(kh)^2 \cdot h = k^2h^3$$

$$A = \sum_{n=0}^{n-1} k^2h^3 = h^3\frac{1}{6}n(n-1)(2n-1) \quad \text{(sum of squares of natural numbers)}$$

$$n = b/h$$

$$A = \frac{b^3}{n^3} \cdot \frac{1}{6}n(n-1)(2n-1)$$

$$= \frac{b^3}{6n^2}(n-1)(2n-1)$$

$$= \frac{b^3}{6}\left(1-\frac{1}{n}\right)(2-1/n)$$

$$\text{as } n \to \infty \quad A \to \frac{b^3}{3} \quad \therefore I = \frac{b^3}{3}.$$

3.3 Double Integrals

For an integral involving two variables - a double integral $f(x,y)$ to be integrated with respect to x and y between certain limits. These limits may represent a closed curve C bounding a region R in the xy-plane. In a similiar way to what we did with single integrals lets divide up the region R into N subregions ΔR_p of are $\Delta A_p, p = 1, 2, \dots N$ and let (x_p, y_p) be any point in subregion ΔR_1. Now consider the sum

$$S = \sum_{p=1}^{N} f\left(x_p, y_p\right)\Delta A_p$$

and let $N \to \infty$ as each of the areas $\Delta A, \to 0$. If the sum S tends to a unique limit I then the is called the double integral of $f(x,y)$ over the region R and is written

$$I = \int_R f(x,y)dA$$

Where dA stands for the element of area in the xy-plane. By choosing the subregions to be small rectangles each of are $\Delta A = \Delta x \Delta y$ and letting both Δx and $\Delta y \to 0$ we can also write the integral as

$$I = \iint_R f(x,y)\,dxdy$$

where we here written out the area explicitly as the product of the coordinate differentials.

Note: Some authors use one integral no matter how many dimensions, some use one for each dimension. We will use the number of differentials explicitly written.

We can solve the integral above in more than one way - we can integrate over y first and then x or we can integrate over x first and then y. If we do the first option we treat x as a constant first and integrate over the y for a specific x. Here is how that is written:

$$I = \int_{x=a}^{x=b} \left(\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\, dy \right) dx$$

The second option is the following:

$$I = \int_{y=ac}^{y=d} \left(\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\, dx \right) dy$$

These two options effectively change the order of the integration.

Question

Evaluate the double integral

$$I = \int \int x^2 y\, dx dy,$$

where R is the triangular region bounded by lines $x = 0$, $y = 0$ and $x + y = 1$.

Answer

At each y integrate over x - this means horizontal chunks:

$$\begin{aligned} I &= \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} x^2 y\, dx dy \\ &= \int_0^1 \left. \frac{x^3}{3} \right|_{x=0}^{x=1-y} y dy \\ &= \int_0^1 \frac{y(1-y)^3}{3} dy \\ &= \left. \frac{y^2}{6} - \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{15} \right|_0^1 \\ &= \frac{1}{60} \end{aligned}$$

Question

Now reverse the order of integration and show that the same result stands.

Answer

At each x integrate over y - this is vertical chunks.

$$\begin{aligned} I &= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} x^2 y \, dy dx \\ &= \int_{x=0}^{x=1} \left. \frac{x^2 y^2}{2} \right|_{y=0}^{y=1-x} dx \\ &= \int_{x=0}^{x=1} \frac{x^2}{2} (1-x)^2 dx \\ &= \int_0^1 \frac{x^2}{2} - x^3 + \frac{x^4}{2} dx \\ &= \left(\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10} \right) \Bigg|_0^1 \\ &= \frac{1}{60} \end{aligned}$$

We can also write a double integral with the differential next to the integral:

$$I = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \, f(x, y)$$

where it is understood that each integral symbol acts on everything to its right and the order of integration is right to left. So $f(x, y)$ is integrated with respect to y first and then x.

When can the order of integrals not be changed?

If the region R is unbounded with some limits infinite or $f(x, y)$ has any discontinuities in R or C.

3.4 Triple Integrals

We can extend the same ideas from double integrals to triple integrals. Divide a region R into subregions ΔR_p and volume ΔV_p where $p = 1, 2, ..., N$. The S is then:

$$= \sum_{p=1}^{N} f(x_p, y_p, z_p)\Delta V_p$$

as $N \to \infty$ $\Delta V_p \to 0$ if S tends to a unique limit I then

$$I = \int_R f(x, y, z)\, dV$$

where dV is an element of volume.

$$\begin{aligned} I &= \int\int\int_R f(x, y, z)\, dxdydz \\ &= \int_{x_1}^{x^2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz\, f(x, y, z) \end{aligned}$$

3.5 Area and Volume

$$A = \int_R dA = \int\int_R dxdy$$

is equal to the area of region R.

The volume under $z = f(x, y)$ is

$$V = \int_R z\, dA = \int\int_R f(x, y)\, dxdy.$$

You can also find the volume directly by using a triple integral and integrating over z.

> **Question**
>
> Find the volume of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Answer

It will help to visualize this if you draw a picture ...

$$
\begin{aligned}
V &= \int \int_R z\, dx dy \\
z &= c(1 - x/a - y/b) \\
V &= \int_0^a dx \int_0^{b(1-x/a)} dy c(1 - x/a - y/b) \\
&= \int_0^a dx c(y - xy/a - y^2/2b)\Big|_0^{b-bx/a} \\
&= abc/6
\end{aligned}
$$

3.6 Masses

4 Problems

4.1 Week 2

Question

The following question can be turned in for formative feedback. Please submit this question on paper at the next problems class.
The equation $3y = z^3 + 3xz$ defines z implicitly as a function of x and y. Evaluate all three second partial derivatives of z with respect to x and/or y. Verify that z is a solution of

$$x\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0.$$

Question

Find the first partial derivatives of the following functions$f(x, y)$: (i) x^2y, (ii) $x^2 + y^2 + 4$, (iii) $\sin(x/y)$
For (i) and (ii) find the the second partial derivatives.

Question

Show that

$$df = x^2\,dy - (y^2 + xy)\,dx$$

is not an exact differential.

Question

The function $f(x, y)$ satisfies the differential equation

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

By changing to new variables $u = x^2 - y^2$ and $v = 2xy$, show that f is, in fact, a functions of $x^2 - y^2$ only.

Tip

Answer In order to change variables we need to understand that u and v depend on both x and y and vice versa so our chain rule for partial derivatives becomes

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial x_j}$$

where we want to change the variables of the partial derivative. Each x_j depends on more than one u_i. In this case we want the partial of f with respect to x and the partial of f with respect to y in terms of u and v.

$$\begin{aligned}
&\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\
&\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\\
&\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = 2y\\
&\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial y} = 2x\\
&\frac{\partial f}{\partial x} = 2x\,\frac{\partial f}{\partial u} + 2y\,\frac{\partial f}{\partial v}\\
&\frac{\partial f}{\partial y} = -2y\,\frac{\partial f}{\partial u} + 2x\,\frac{\partial f}{\partial v}\\
&y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0
\end{aligned}$$

substituting in for the partials of f with respect to x and y in terms of u and v

$$\begin{aligned}
&y\left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right) + x\left(-2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}\right) = 0\\
&2xy\frac{\partial f}{\partial u} - 2xy\frac{\partial f}{\partial u} + 2y^2\frac{\partial f}{\partial v} + 2x^2\frac{\partial f}{\partial v} = 0\\
&2(x^2+y^2)\frac{\partial f}{\partial v} = 0
\end{aligned}$$

Assuming $x^2 + y^2$ is not zero (in other words we are stipulating that x and y are not both zero) this means that $\frac{\partial f}{\partial v} = 0$ so f is then only a function of u which is the same as saying $f(x^2 - y^2)$.

Question

Find the total derivative of $f(x, y) = x^2 + 3xy$ with respect to x given that $y = \sin^{-1} x$. (From the lecture notes on Total Differential and Total Derivative)

Question

Show that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

(From the lecture on Thermodynamic Relations)